Sets & Sequences

CISC 2210 — CUNY Brooklyn College
Lecture Notes

Set: Definition

We begin our course by looking into the simplest discrete structure: a set.

A **set** is an unordered collection of various sorts of objects, which we call **items** or **elements**. Examples:

- $\{1, 2, 3\}$
- $\{A, C, E, G\}$
- $\{\text{Alice}, \text{Carol}, \text{Eugene}, \text{George}\}$
- {😀, 😁, 😎, 😜}

We write the elements of a set between a pair of curly braces, $\{$ and $\}$. In $\LaTeX$, we write:

- `$\{1, 2, 3\}$`
- `$\{A, C, E, G\}$`
- `$\{\text{Alice}, \text{Carol}, \text{Eugene}, \text{George}\}$`

Set: Definition

To create the set of smilies $\{$😀, 😁, 😎, 😜$\}$ in $\LaTeX$, first import the package

```
\usepackage{twemojis}
```

and then use the following 2 statements in the body of the document:

```
\setlength{\twemojiDefaultHeight}{0.5cm} % Increase the size of the emojies.

$\{\twemoji{grinning}, \twemoji{grin}, \twemoji{smiling face with sunglasses}, \twemoji{winking face with tongue}\}$
```

Tons of more smilies and emojis are listed in the following PDF documentation of the `twemojis` package: https://mirror.las.iastate.edu/tex-archive/macros/latex/contrib/twemojis/twemojis.pdf.

Set: Definition

As we mentioned, sets are unordered collections. This means that changing the order of the listed elements or repeating any of the elements in a set doesn't change the set: it's still the same set.

Example: All the 3 sets

- $\{1, 2, 3\}$
- $\{3, 1, 2\}$, and
- $\{1, 2, 2, 3, 1, 3\}$

are just different ways to write out the same exact set: $\{1, 2, 3\}$.

We say that all these set are **equal** to one another (in $\LaTeX$: `$=$`):

$$\{1, 2, 3\} = \{3, 1, 2\} = \{1, 2, 2, 3, 1, 3\}$$

Set: Definition

Another example of equal sets:

{😀, 😁, 😎} = {😀, 😎, 😁, 😎, 😀, 😀, 😀}.

The following two sets, however, aren't equal (***why so?***), so we use the symbol '$\neq$' (in LaTeX: `$\neq$`):

{😀, 😎, 😁, 😎, 😀, 😀, 😀} ≠ {😀, 😎, 😁, 😎, 😜, 😀, 😀}

Fun Class Activity: For each pair of sets below, tell if they are equal or not:

1. $\{A, B, C\}$ and $\{C, A, B\}$
2. $\{1, 10, 100\}$ and $\{10, 100, 100, 1, 10\}$
3. $\{\text{Alice}, \text{Bob}, \text{Carol}\}$ and $\{\text{Alice}, \text{Bob}, \text{Mel}\}$

Sequence: Definition

Another discrete structure we'll talk about in this topic is a sequence.

A **sequence** is an <u>ordered</u> collection of various sorts of objects, which we call **terms** or sometimes **coordinates**, but the names **items** or **elements** are also proper. <u>Examples</u>:

- $(1, 2, 3)$ or $\langle 1, 2, 3 \rangle$ [`$\langle 1, 2, 3\rangle$`]
- $(1, 2, 3, 4, \ldots)$ or $\langle 1, 2, 3, 4, \ldots \rangle$
- $(\text{Alice, Carol, Eugene, George})$ or $\langle \text{Alice, Carol, Eugene, George} \rangle$
- $(😀, 😁, 😎, 😜)$ or $\langle 😀, 😁, 😎, 😜 \rangle$

Unlike sets, since sequences are ordered, changing the order or repeating a term creates a whole new sequence:

$$\langle 😀, 😁, 😎 \rangle \neq \langle 😀, 😎, 😁, 😎, 😀, 😀, 😀 \rangle.$$

We will get back to sequences at the end of this topic. Let's get back to sets now.

Sequence: Definition

Any questions so far?

Ways to Write Out Sets

There are 4 general ways of writing (expressing) the elements of a set:

1. Listing all the elements (called the **roster method**,) such as in $\{A, B, C\}$. Good only if the set is finite and short.
2. Using the ellipsis $\dots$ (`$\dots$`) such as in $\{1, 4, 9, 16, \dots\}$ or in $\{1, 2, \dots, 100\}$ for long finite sets or infinite sets.
3. Writing a special list's symbol. Here are some special lists:
   - $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of all non-negative integers. (`$\mathbb{N}$`) ($\mathbb{N}$ is for 'Natural numbers'.)
   - $\mathbb{P} = \{1, 2, 3, \dots\}$, the set of all positive integers. (`$\mathbb{P}$`)
   - $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, the set of all integers. (`$\mathbb{Z}$`) (*Zahl* means 'whole' in German.)
   - $\mathbb{Q} = \frac{m}{n}$, where m and n are integers: the set of all rational numbers. (`$\mathbb{Q}$`) (From the word 'Quotient'.)
   - $\mathbb{R}$, the set of all real numbers. (`$\mathbb{R}$`), and
   - $\mathbb{C}$, the set of all complex. (`$\mathbb{C}$`).
4. Using a **set rule notation** (also called the **set builder notation**) as in $\{n \mid n \text{ is in } \mathbb{P} \text{ and } n > 10\}$, which is read as "all numbers n such that n is in $\mathbb{P}$ and is greater than 10." More on this on the next slide.

Ways to Write Out Sets

4. As we saw on the previous slide, the set rule notation consists of two parts within the curly braces: the part *before* the $|$ symbol, which is the **format** of the elements in the set, and the part *after* the $|$ symbol, which imposes **rules** on the elements of the set. The $|$ symbol, which is called a 'bar' and is written in $\LaTeX$ as `$|$`, is read in English as "such that".

   You can include as many rules as you want; also, they can be separated with commas, like in: $\{n \mid n \text{ is in } \mathbb{P}, n > 10\}$. Here are some examples and their meaning in math and English:
   - $\{n \mid n \text{ is in } \mathbb{P}, n > 10\} = \{11, 12, 13, \ldots\}$, "the set of all n such that n is a positive integer, and n is greater than 10", or "the set of all integers greater than 10", shortly.
   - $\{n^2 \mid n \text{ is in } \mathbb{N}\} = \{0, 1, 4, 9, 16, \ldots\}$, "the set of all numbers n^2 such that n is a non-negative integer," or "the set of integer squares", shortly.
   - $\{x^2 + 2x + 1 \mid x \text{ is in } \mathbb{R}\}$, "the set of all numbers $x^2 + 2x + 1$ such that x is a real number." This is the set of all values that the real quadratic function $x^2 + 2x + 1$ outputs.

Ways to Write Out Sets

Fun Class Activity:

1. List all the elements of the set $S = \{2n \mid 0 \le n \le 5\}$. Assume that n is a non-negative integer.
2. Express the set $S = \{2n \mid 0 \le n \le 5\}$ using the ellipsis notation.
3. Express the set $\mathbb{Z}$ in set rule notation.
4. Express the set $\mathbb{Z}$ using the ellipsis notation.

Fun facts: The 'less than or equal to' symbol, or $\le$, is written in $\LaTeX$ as `$\le$`, and the 'greater than or equal to' symbol, or $\ge$, is written in $\LaTeX$ as `$\ge$`.

So $S = \{2n \mid 0 \le n \le 5\}$ is written as `$S = \{2n\ |\ 0 \le n \le 5\}$`.

The spaces that we add around the $\mid$ symbol (`$\ |\ $`) prevent the notation from being squashed, as in $S = \{2n|0 \le n \le 5\}$ and `$S = \{2n | 0 \le n \le 5\}$`.

Element Inclusion vs. Subsets

When an element n is a part of a set S, we write: $n \in S$ (`$n \in S$`), which is read as "n is in S", "n belongs to S", or "n is a part of S".

When an element n is <u>not</u> in the set S, we write: $n \notin S$ (`$n \not\in S$`).

<u>Examples</u>: Let $S =$ {😀, 😁, 😎, 😜}. We see that 😁 $\in S$, while 🤗 $\notin S$.

When you have two sets, A and B, and all the elements of A are included in B, we say that A is a **subset** of B. We write: $A \subseteq B$ (`$A \subseteq B$`). Also, if B has more unique element than A, we say that A is a **proper subset** of B, and write: $A \subset B$ (`$A \subset B$`).

If at least one of the items of A is not in B, then A is not a subset of B, so we write $A \not\subseteq B$ or $A \not\subset B$ (`$A \not\subseteq B$` or `$A \not\subset B$`).

Element Inclusion vs. Subsets

Examples: Let $A = \{a, b, c, d\}$, $B = \{a, b, d\}$, and $C = \{a, b, z\}$. We see that $B \subset A$, while $C \not\subset A$. (***Is A a subset of B? Why or why not?***)

Now that we know what subsets are, we can officially define what set equality is: if A and B are sets such that $A \subseteq B$ and $B \subseteq A$, it must hold that $A = B$.

One additional special set that we should know about is the empty set, written either as $\emptyset$ (`$\emptyset$`) or as $\{\}$ (`$\{\}$`).

Fun facts: If a set S doesn't contain $\emptyset$ as one of its elements, then $\emptyset \notin S$. However, $\emptyset \subset S$: this is true for any set S whatsoever.

Also, $\emptyset \subseteq \emptyset$, and $S \subseteq S$ for any set S.

Element Inclusion vs. Subsets

Fun Class Activity: Let $A = \{$😂, 😆, 🤓$\}$ and $B = \{$😂, 🤓$\}$. Determine which of the following statements are true:

1. 😆 $\in A$
2. 😆 $\in B$
3. $\{$😆$\} \in A$
4. $\{$😆$\} \subset A$
5. $B \subset A$
6. $A \subset B$
7. $\emptyset \in B$
8. $\emptyset \subseteq B$
9. $\emptyset \subset B$
10. $B \subset B$

Element Inclusion vs. Subsets

Any questions so far?

Set Cardinality

The **cardinality** of a set is the number of elements that the set has, i.e., the size of the set. For a set S, we denote its cardinality as $|S|$ (`$|S|$`). Examples:

- For $S = \{$😀, 😁, 😎, 😜$\}$, we have $|S| = 4$.
- For $N = \{1, 2, \ldots, 100\}$, we have $|N| = 100$.
- For $I = \{1, 2, 3, \ldots\}$, we have $|I| = \infty$ (`$\infty$`).

While the cardinality of a finite set is a fixed, nonnegative integer, this of an infinite set is infinity. There are actually different types of infinity, such as **countable** and **uncountable**; you'll learn more on those in more advanced CS courses.

Fun Class Activity: What is each of the following equal to?

1. $|\mathbb{Z}|$
2. $|\{2n \mid 0 \leq n \leq 5\}|$
3. $|\emptyset|$

Power Set

So far, we looked into sets that contain various objects. Now, we'll also talk about sets whose elements are other sets!

The **power set** of a set S is a set containing all the possible subsets of S. It is a sort of a multiverse of subsets. We denote the power set as $\mathcal{P}(S)$ (`$\mathcal{P}(S)$`). Examples:

- For $S = \{😀, 😁\}$, we have $\mathcal{P}(S) = \{\emptyset, \{😀\}, \{😁\}, \{😀, 😁\}\}$.
- For $N = \{1, 2, 3\}$, we have $\mathcal{P}(N) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- For $A = \{\text{Alice}\}$, we have $\mathcal{P}(A) = \{\emptyset, \{\text{Alice}\}\}$.

If a set S contains n elements, how many elements does $\mathcal{P}(S)$ have?

From the above examples, we see that (1) when $|S| = 1$, $|\mathcal{P}(S)| = 2$, (2) when $|S| = 2$, $|\mathcal{P}(S)| = 4$, and (3) when $|S| = 3$, $|\mathcal{P}(S)| = 8$. Can you see a pattern? What is the formula for $|\mathcal{P}(S)|$ in the general case of $|S| = n$?

Power Set

Fun Class Activities:

1. For each of the following sets, indicate (1) the cardinality of the given set, (2) the cardinality of the powerset for that set, and (3) the powerset items themselves in a listed form.
    1. $L = \{X, Y\}$
    2. $S = \{$😊, 😃, 😂$\}$
    3. $E = \{\}$ (Remember that this is just another way to write $\emptyset$: the empty set.)
2. What is the cardinality of the powerset of each of the following sets?
    1. $N = \{0, 1, 2, \ldots, 100\}$
    2. $T = \{n \mid n \in \mathbb{N}, 0 \leq n \leq 30, n \text{ is odd}\}$
    3. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Cartesian Product

Just as we can have a set of sets, we can also have a set of sequences.

The **Cartesian product** of sets A and B is a set containing all the possible **ordered pairs** (points) of the form (a, b), in which $a \in A$ and $b \in B$. In other words, it is the set of sequences of 2 elements, such that the first coordinate is an element in the first set (A), and the 2nd coordinate is an element in the second set (B). The notation of the Cartesian product is $A \times B$ (`$A \times B$`). Examples:

- For $A = \{0, 1, 2\}$ and $B = \{8, 9\}$, we have $A \times B = \{(0, 8), (0, 9), (1, 8), (1, 9), (2, 8), (2, 9)\}$.
- For $S = \{1, 2, 3\}$, we have $S \times S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

If $|\boldsymbol{A}| = \boldsymbol{n}$ and $|\boldsymbol{B}| = \boldsymbol{m}$, then how much is $|\boldsymbol{A} \times \boldsymbol{B}|$ equal to? Can you spot a pattern in the examples above?

Cartesian Product

Fun Class Activity: Consider the sets $A = \{0, 1, 2\}$, $B = \{0, 2\}$, and $C = \{1, 2\}$. Answer the following questions:

1. Is the following statement true: $\{(0, 1), (2, 2)\} \subset A \times B$?
2. Is the following statement true: $\{(0, 1), (2, 2)\} \subset A \times C$?
3. What is $|A \times B|$?
4. List all the elements in $A \times B$.
5. List all the elements in $A \times C$.
6. What is $|B \times C|$?
7. List all the elements in $B \times C$.
8. Is $\emptyset \in A \times B$?
9. Is $\emptyset \subset A \times B$?

Any questions so far?

Set Operations

We can perform several mixing and matching operations on sets to get new sets:

The **intersection** of two sets A and B, denoted as $A \cap B$ (`$A \cap B$`), is the set of elements that are in both A and B. Formally: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. Examples:

- If $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$, then $A \cap B = \{3,4\}$.
- If $A = \{a,b,c\}$ and $B = \{c,d,e\}$, then $A \cap B = \{c\}$.
- If $A = \{1,2,3\}$ and $B = \{4,5,6\}$, then $A \cap B = \emptyset$.

Two sets A and B are **disjoint** if they have no elements in common, i.e., $A \cap B = \emptyset$. Examples:

- If $A = \{1,2,3\}$ and $B = \{4,5,6\}$, then A and B are disjoint.
- If $A = \{1,2,3\}$ and $B = \{3,4,5\}$, then A and B are *not* disjoint since $A \cap B = \{3\}$.

Set Operations

The **union** of two sets A and B, denoted as $A \cup B$ (`$A \cup B$`) is the set containing all elements from A or B (or both): $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. Examples:

- If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.
- If $A = \{x, y\}$ and $B = \{y, z\}$, then $A \cup B = \{x, y, z\}$.
- If $A = \{1, 2\}$ and $B = \{3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

Fun Class Activities:

1. Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 6, 7, 10\}$. Find $A \cap B$ and $A \cup B$
2. Given the sets $X = \{a, b, c, d\}$ and $Y = \{c, d, e, f\}$, find $X \cap Y$ and $X \cup Y$.
3. Let $P = \{10, 20, 30, 40\}$ and $Q = \{30, 40, 50, 60\}$. Compute $P \cap Q$ and $P \cup Q$.

Set Operations

The **universal set**, denoted by U, is the set that contains all elements under consideration (that's not the set of everything, though.) Every set is a subset of U: $A \subseteq U$. Examples:

- If $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3\}$, then $A \subseteq U$.
- If $U = \mathbb{R}$ (the set of real numbers), then $\mathbb{N}$ (natural numbers) is a subset of U.
- If $U = \{a, b, c, d, e\}$ and $B = \{a, d, e\}$, then $B \subseteq U$.

The **complement** of a set A, denoted by A^c (`$A^c$`) or $\overline{A}$ (`$\overline{A}$`), consists of all elements in the universal set U that are not in A: $A^c = \{x \mid x \in U \text{ and } x \notin A\}$. Examples:

- If $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$, then $A^c = \{2, 4\}$.
- If $U = \mathbb{Z}$ and A is the set of even integers, then A^c is the set of odd integers.
- If $U = \{a, b, c, d\}$ and $A = \{a, c\}$, then $A^c = \{b, d\}$.

Set Operations

The **relative complement** (also: **set difference**) of A in B, denoted by as $B - A$ or $B \setminus A$ (`$B \setminus A$`), is the set of elements in B that are not in A: $B - A = \{x \mid x \in B \text{ and } x \notin A\}$. Examples:

- If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then $B - A = \{4, 5\}$.
- If $A = \{a, b\}$ and $B = \{a, b, c, d\}$, then $B - A = \{c, d\}$.
- If $A = \{1, 2\}$ and $B = \{1, 2\}$, then $B - A = \emptyset$.

The **symmetric difference** of two sets A and B, denoted by $A \triangle B$ (`$A \triangle B$`) is the set of elements in either A or B, but not in both: $A \triangle B = (A - B) \cup (B - A)$. Examples:

- If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \triangle B = \{1, 2, 4, 5\}$.
- If $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$, then $A \triangle B = \{a, d, e\}$.
- If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $A \triangle B = \emptyset$.

Set Operations

Fun Class Activity: Let the universal set be

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

and the following sets be defined as: $A = \{1, 3, 5, 7, 9\}, \quad B = \{2, 3, 6, 7, 10\}, \quad C = \{4, 5, 6, 7\}$.

Compute the following:

1. $A \cap B$
2. $A \cup C$
3. A^c
4. $B - C$
5. $A \triangle B$

Any questions so far?

Set Operations: Venn Diagrams

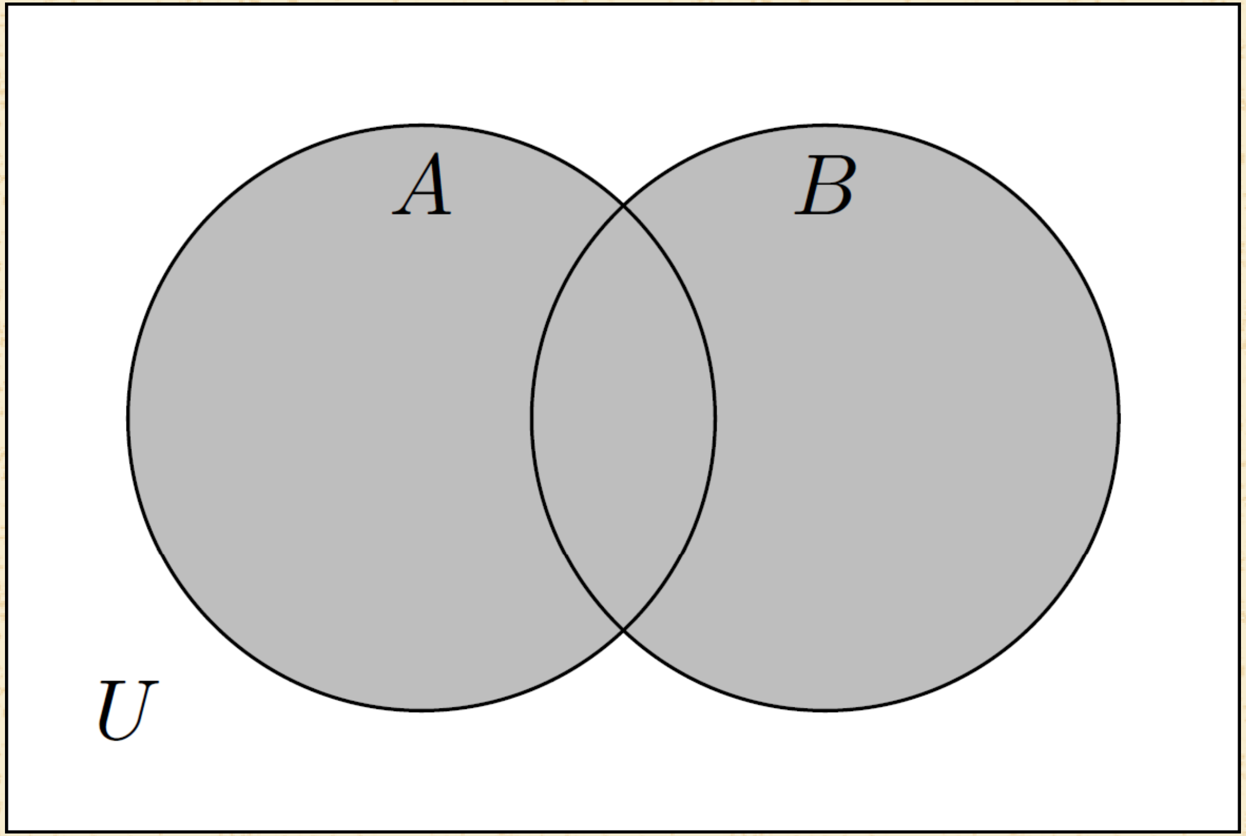
We can illustrate these set operations using **Venn diagrams**, which are 2D diagrams featuring a circle for each discussed set.

Not only do the following slides include Venn diagrams for the operations we presented, but also the $\LaTeX$ code that you can use to generate them on your own!

The $\LaTeX$ library that we use to create these diagrams is called `venndiagram` and is used for a quick sketching of Venn diagrams for 2 sets or 3 sets. More information about using this library, as well as examples, is in the guide on `venndiagram` at https://us.mirrors.cicku.me/ctan/macros/latex/contrib/venndiagram/venndiagram.pdf

One more note before we proceed to show the diagrams: the document type that we used in the following code, `standalone`, is used for the creation of PDF files as large as the diagram's size. In other words, we aren't creating Letter-sized or A4-sized pages; we are creating a small PDF page just to feature that standalone diagram.

Set Operations: Venn Diagrams



A generic diagram on two sets, A and B, whose areas (circles) are filled. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

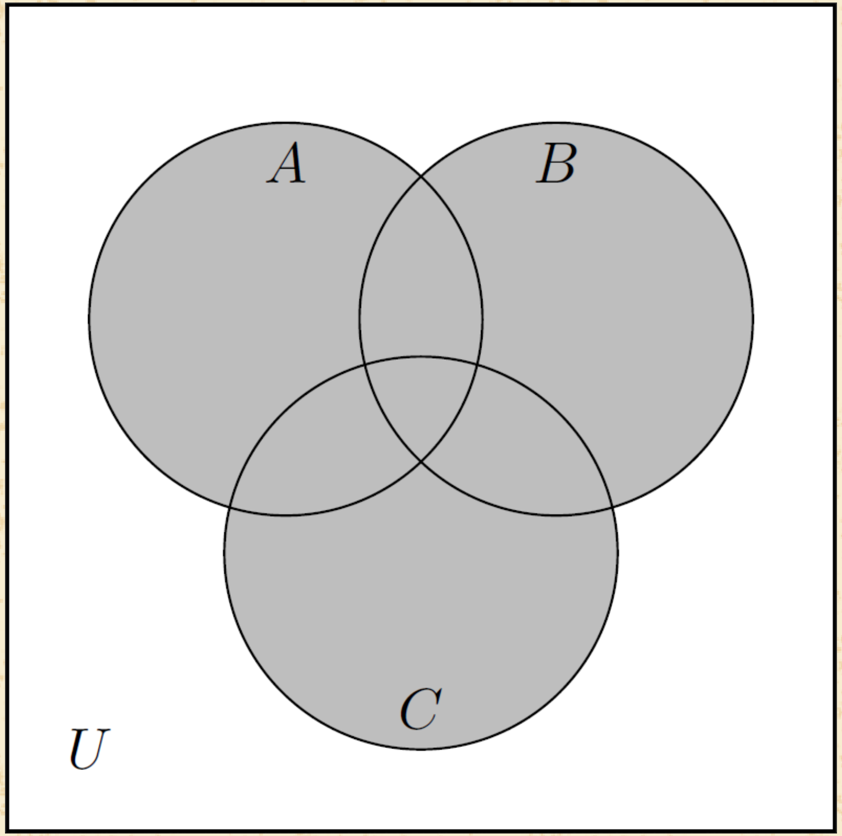
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[labelNotAB={$U$}]
\fillA \fillB
\end{venndiagram2sets}%
\end{document}
```

Set Operations: Venn Diagrams



A generic diagram on three sets, A, B, and C, whose areas (circles) are filled. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

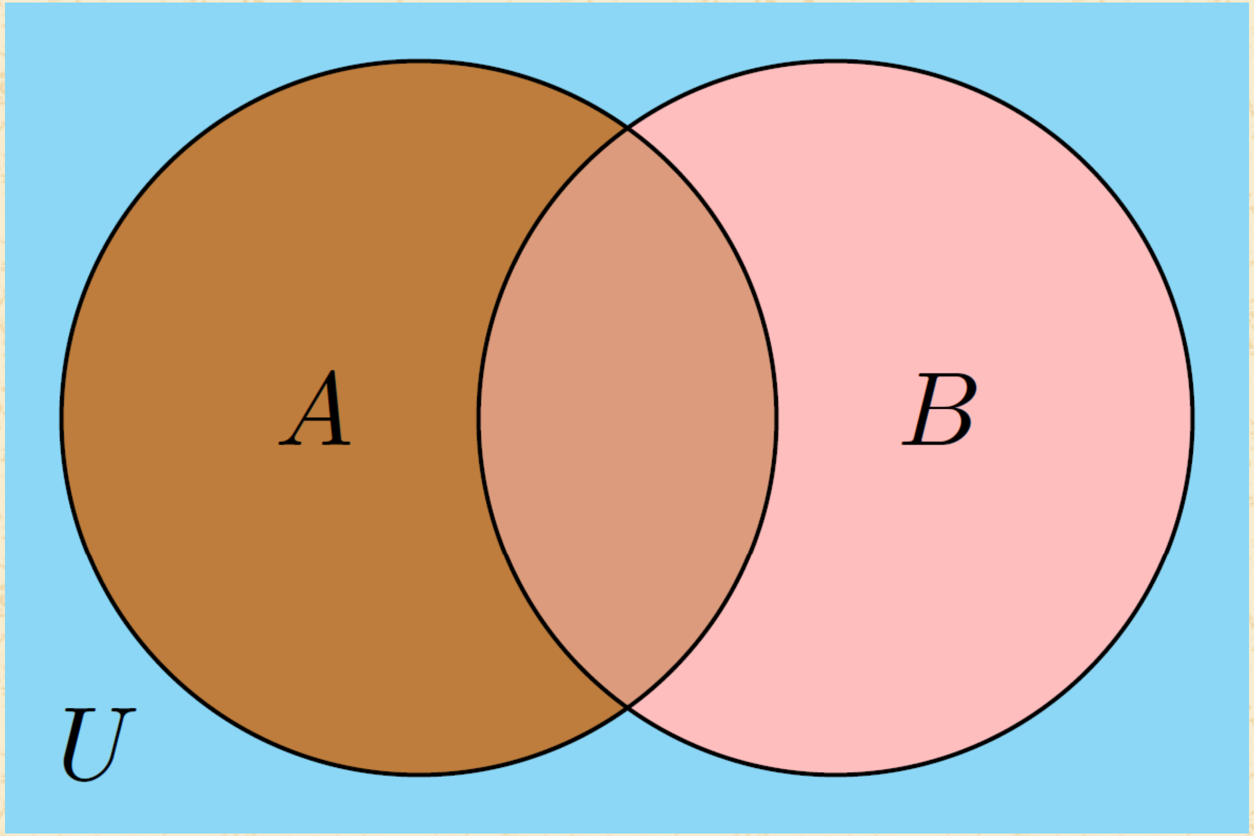
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram3sets}[labelNotABC={$U$}]
\fillA \fillB \fillC
\end{venndiagram3sets}%
\end{document}
```

Set Operations: Venn Diagrams



A customized diagram on two sets, A and B. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

The code that generates the previous PDF diagram:

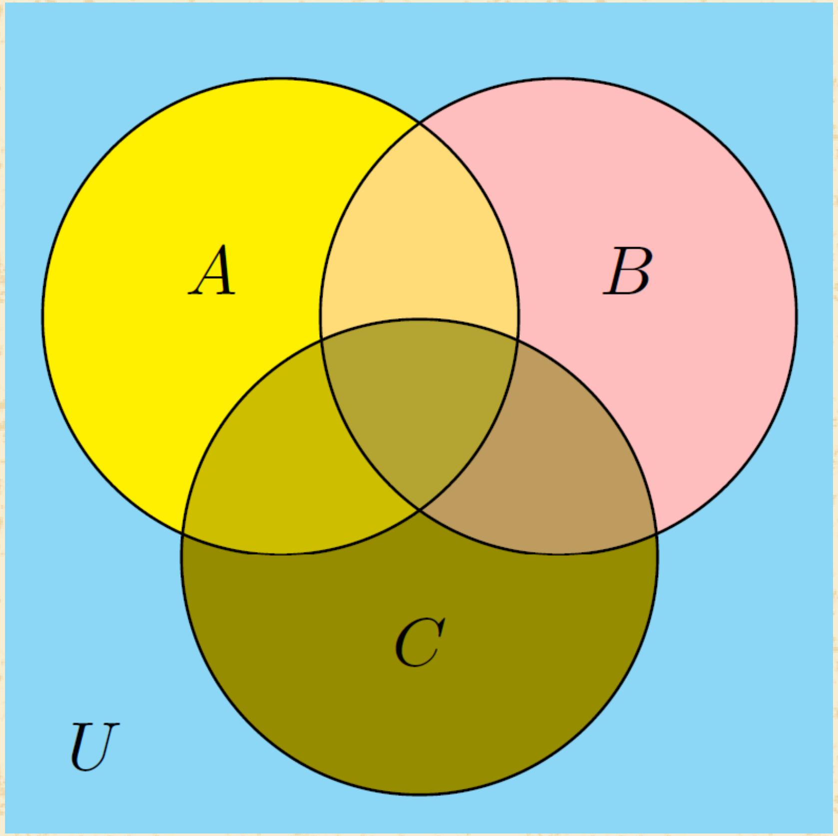
```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

% ---------------------------------

% New Colors:
\colorlet{first}{brown}
\colorlet{second}{pink}
\colorlet{rest}{cyan!40} % 40% cyan and 60% white
\begin{document}
```

Set Operations: Venn Diagrams



A customized diagram on three sets, A, B, and C. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

The code that generates the previous PDF diagram:

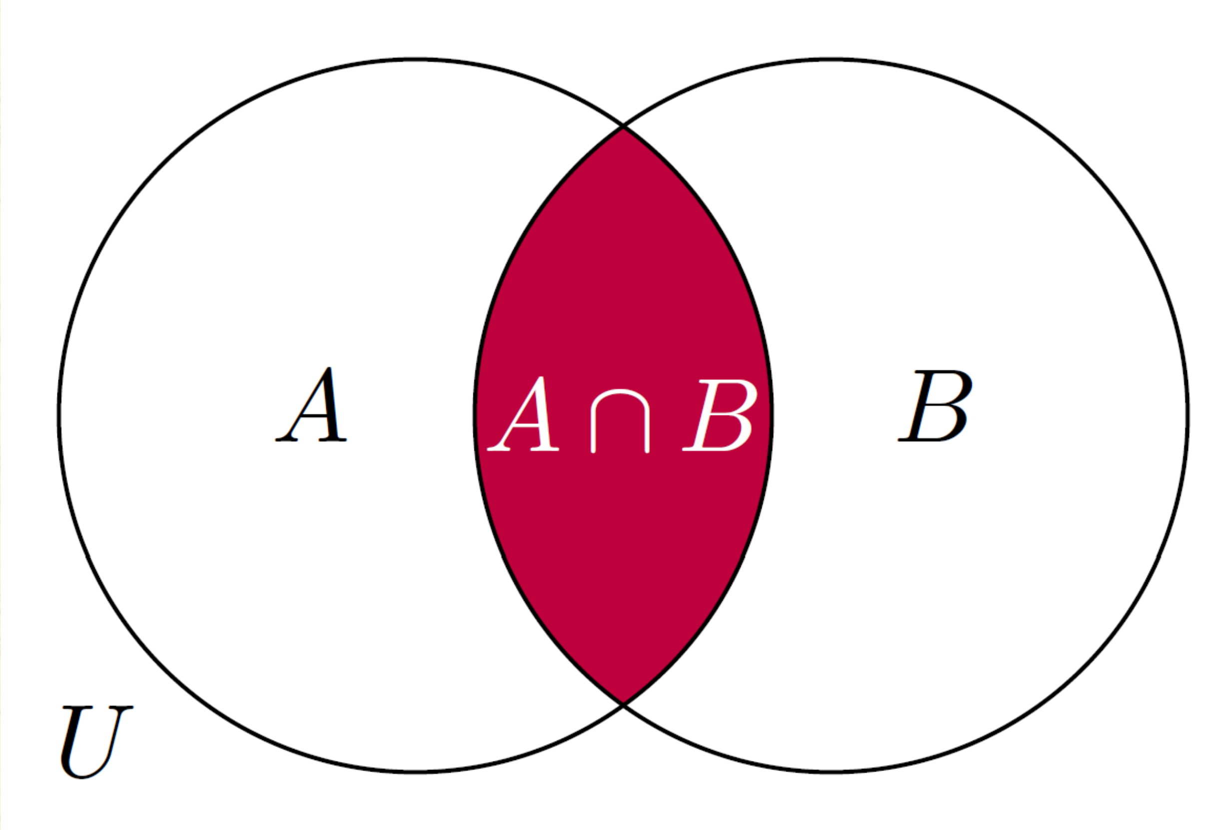
```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

% ---------------------------------

% New Colors:
\colorlet{first}{yellow}
\colorlet{second}{pink}
\colorlet{third}{olive}
\colorlet{rest}{cyan!40} % 40% cyan and 60% white
```

Set Operations: Venn Diagrams



The area representing $A \cap B$ (intersection) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

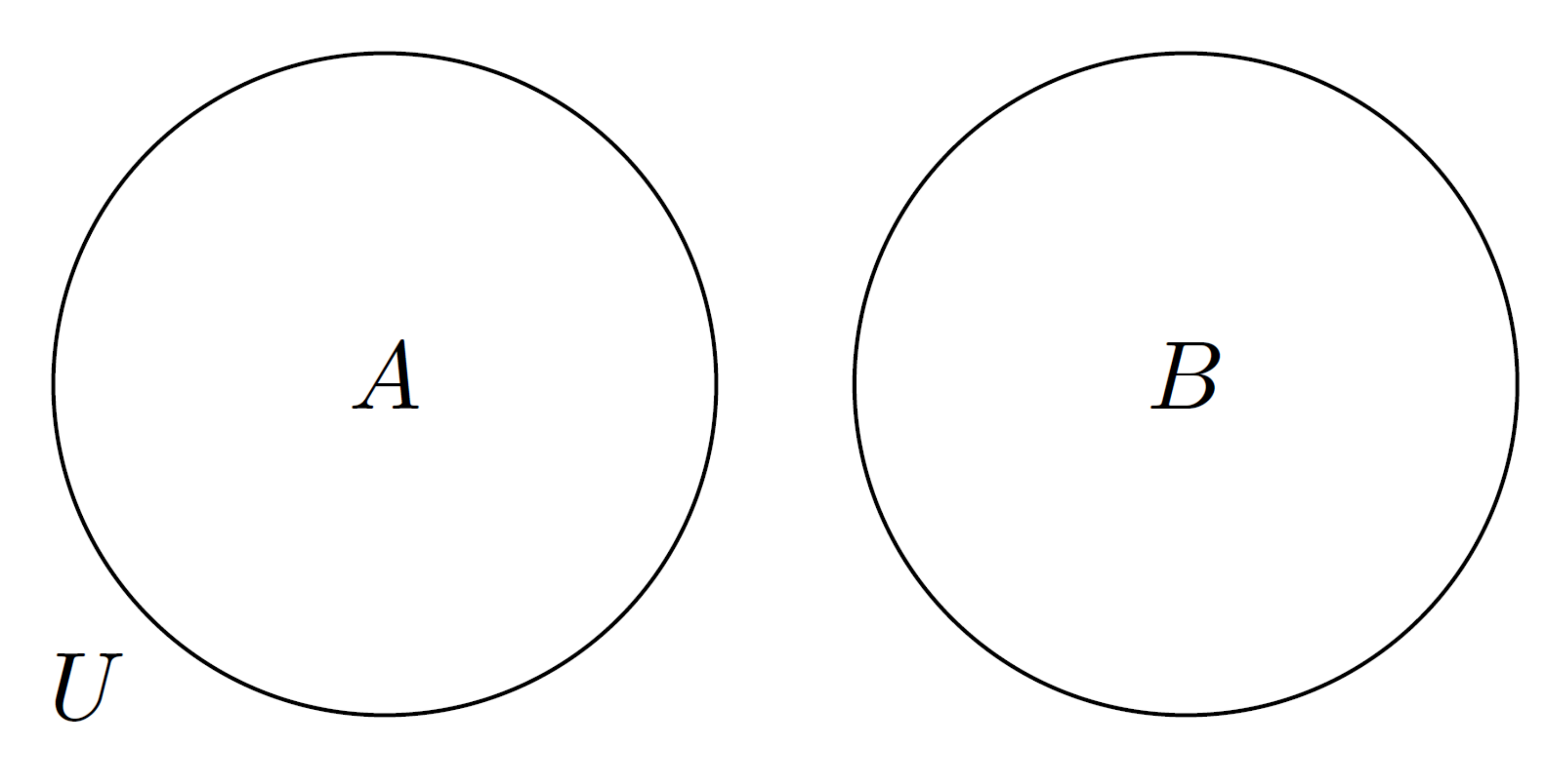
The code that generates the previous PDF diagram:

```latex
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={}, overlap=1cm, shade=purple, labelAB={\color{white} $A \cap B$}]
\fillACapB % The intersection of A and B
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



Here, since A and B don't intersect, we have $A \cap B = \emptyset$. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

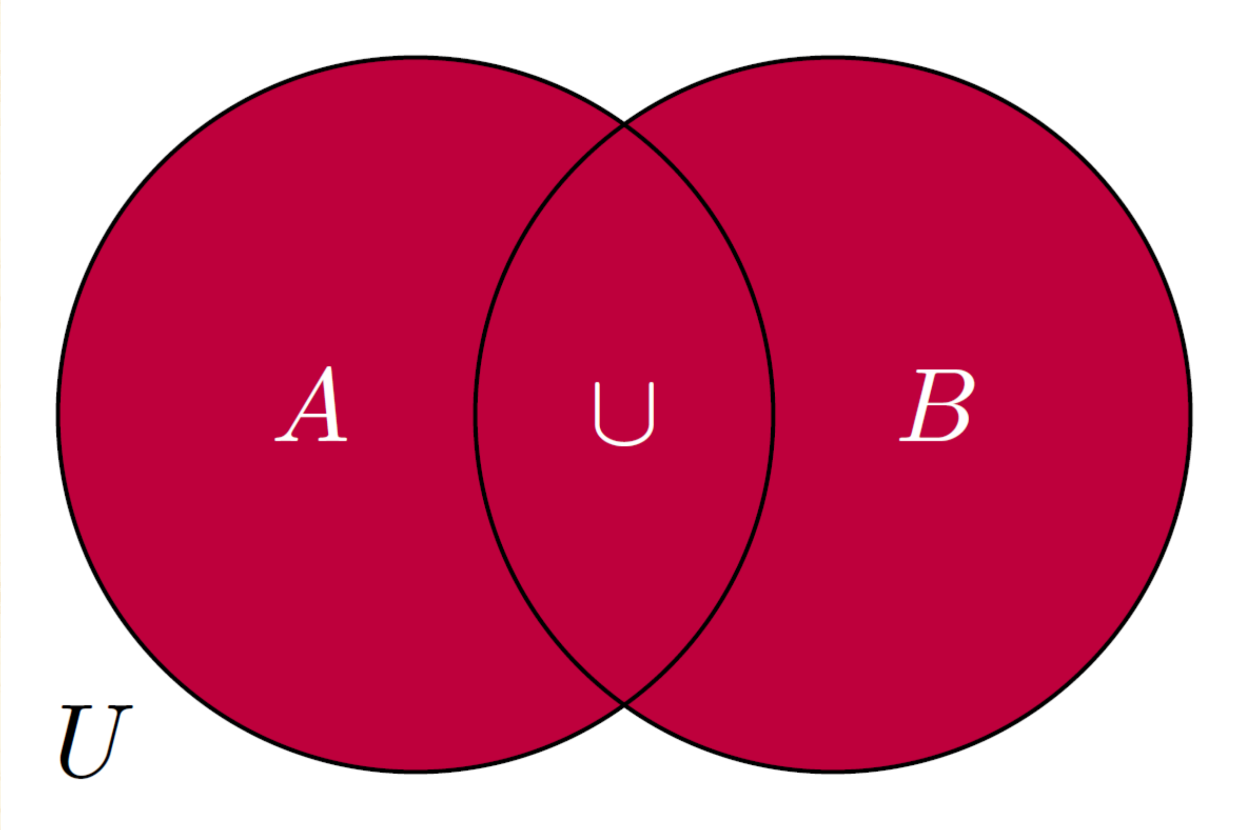
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={}, overlap=-0.5cm, shade=purple]
\fillACapB % There's no intersection for A and B!
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $A \cup B$ (union) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

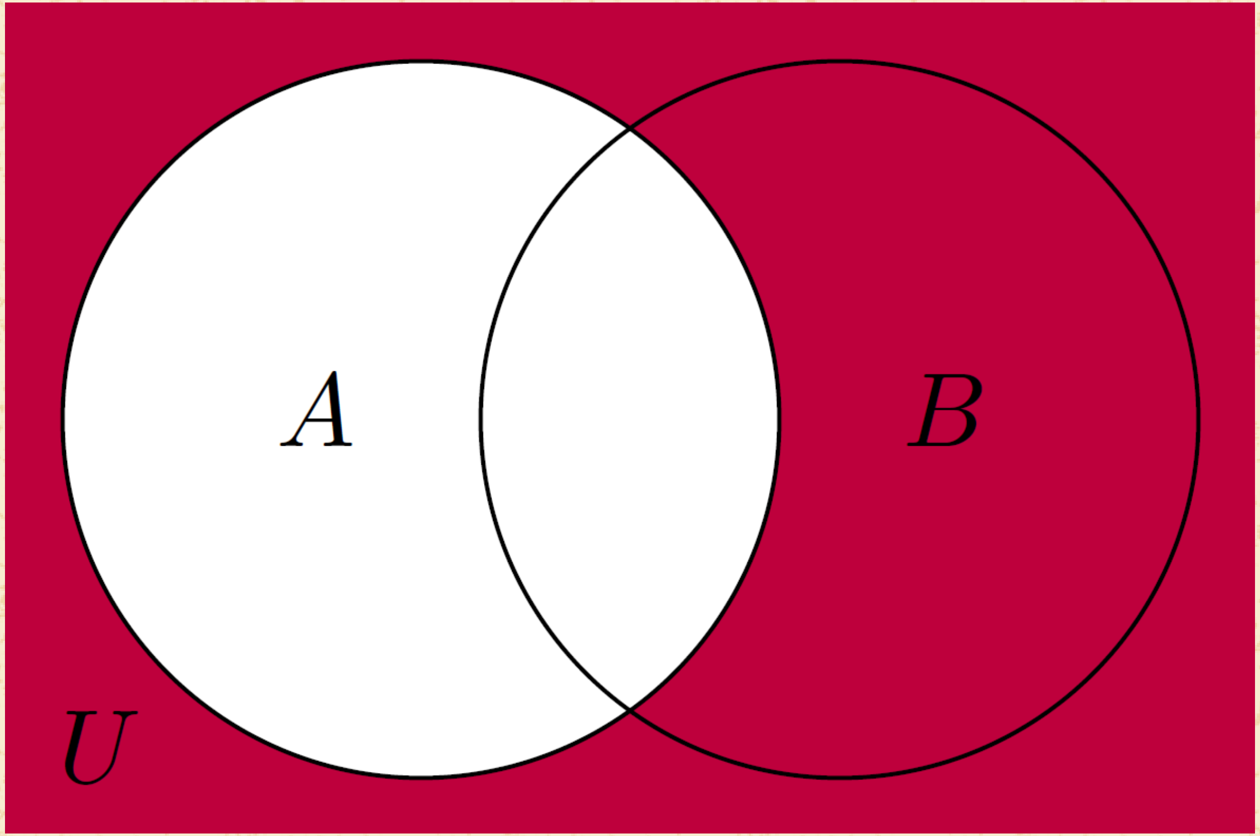
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={}, overlap=1cm, shade=purple, labelAB={\color{white} $\cup$}]
\fillA \fillB % The union of A and B
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $A^c = \overline{A} = U \setminus A = U - A$ (A complement) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

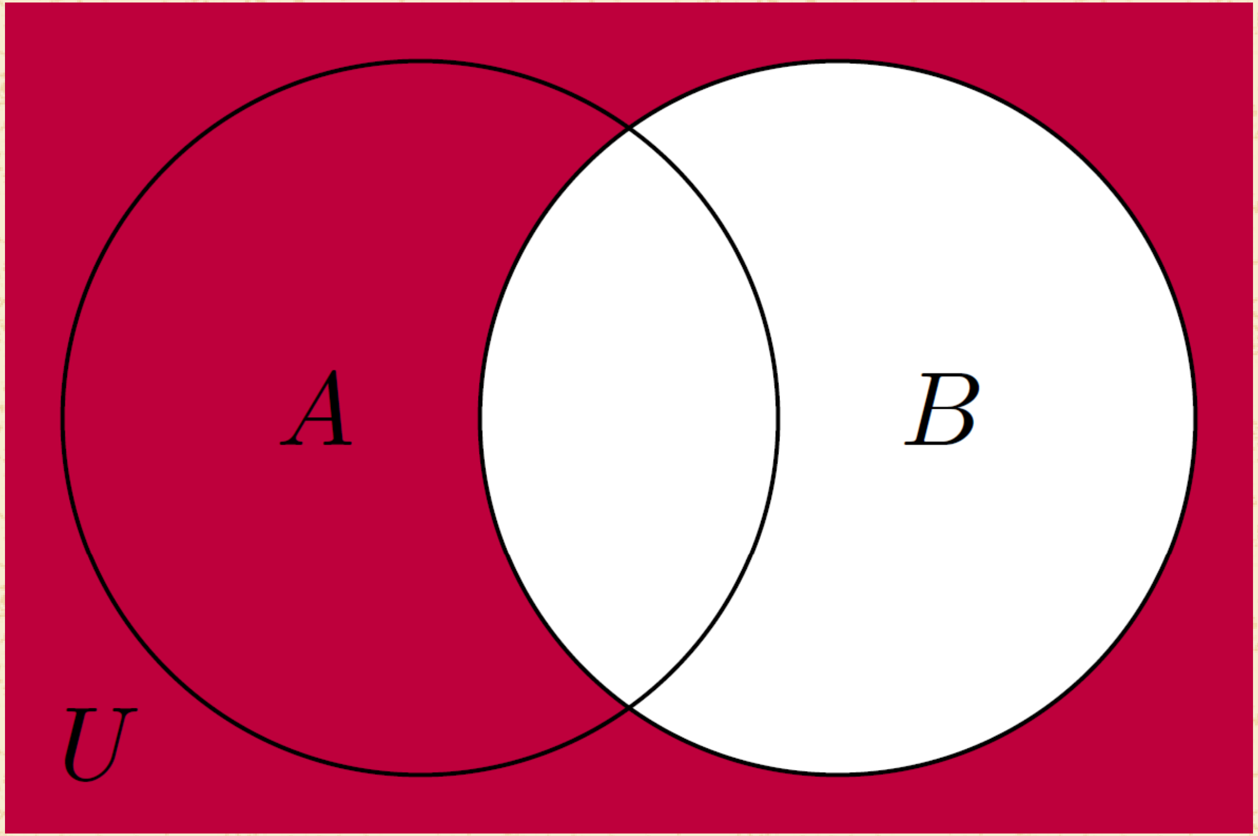
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={}, overlap=1cm, shade=purple]
\fillNotA % The complement of A (= U \ A = U - A)
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $B^c = \overline{B} = U \setminus B = U - B$ (B complement) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

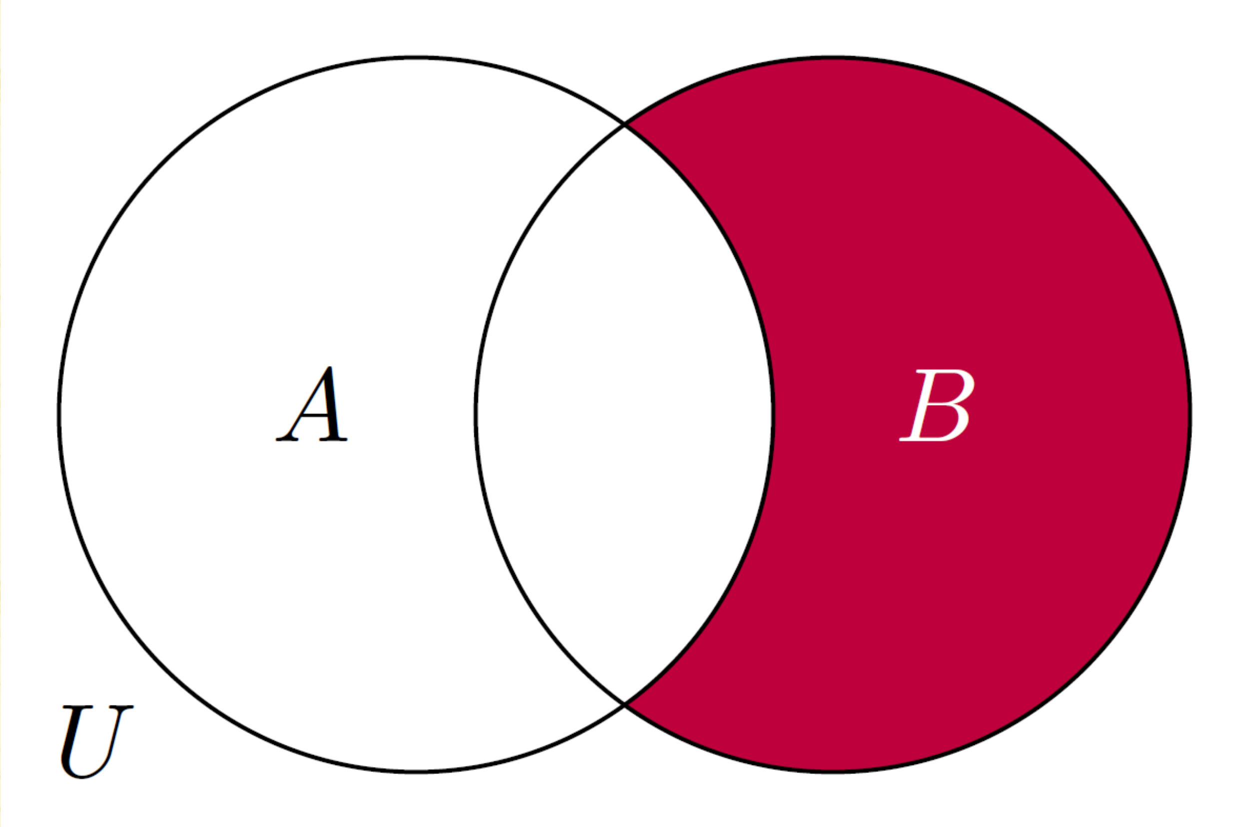
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={},
overlap=1cm, shade=purple]
\fillNotB % The complement of B (= U \ B = U - B)
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $B \setminus A = B - A$ (Relative Complement $B - A$) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

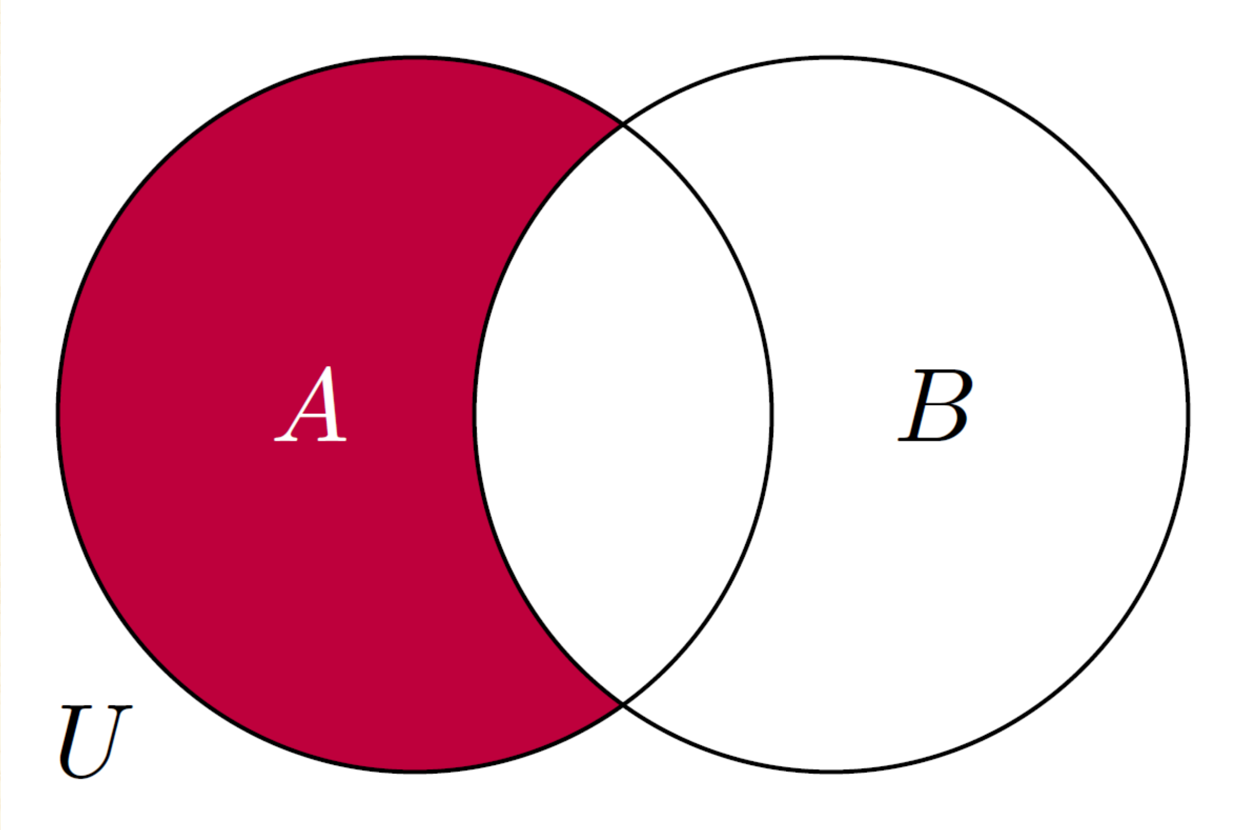
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={},
overlap=1cm, shade=purple]
\fillBNotA % The set B \ A = B - A
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $A \setminus B = A - B$ (Relative Complement $A - B$) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

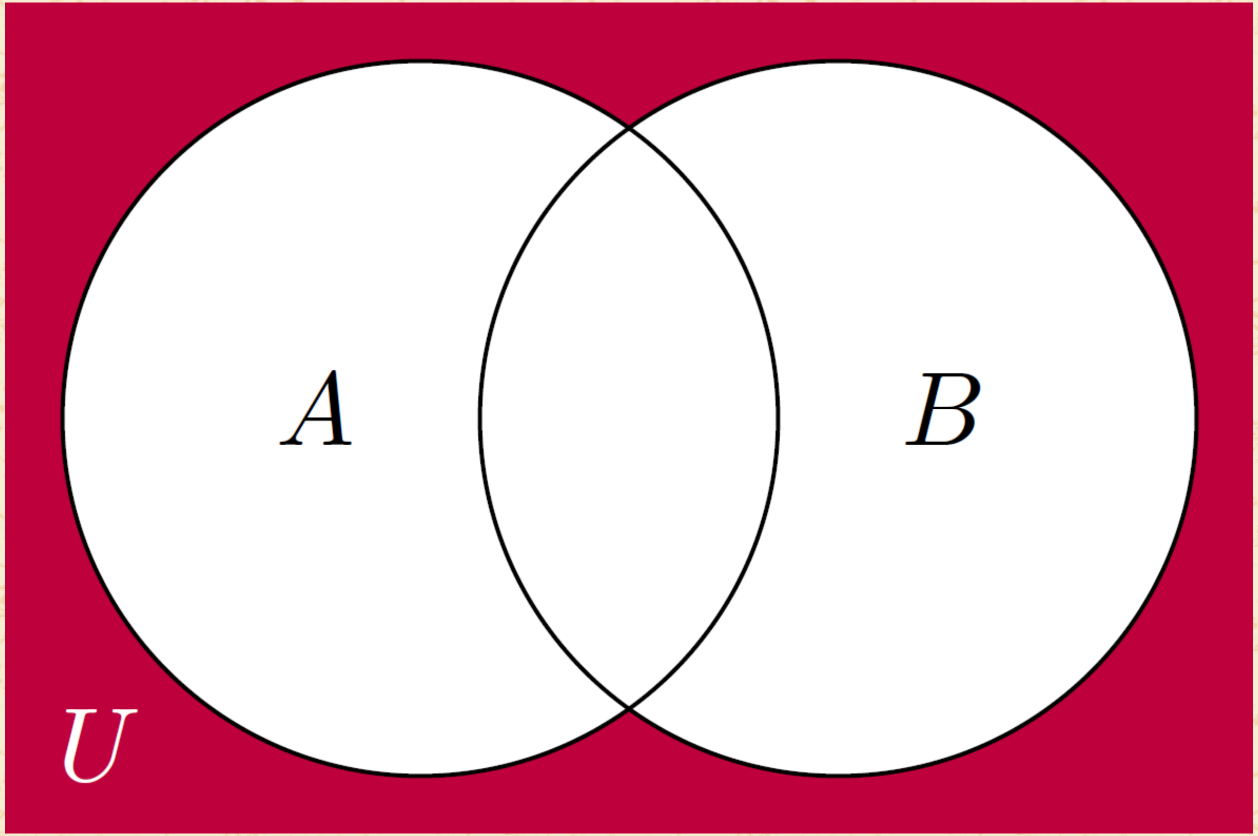
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={},
overlap=1cm, shade=purple]
\fillANotB % The set A \ B = A - B
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $U \setminus (A \cup B) = U - A - B$ (Relative Complement $U - A - B$) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

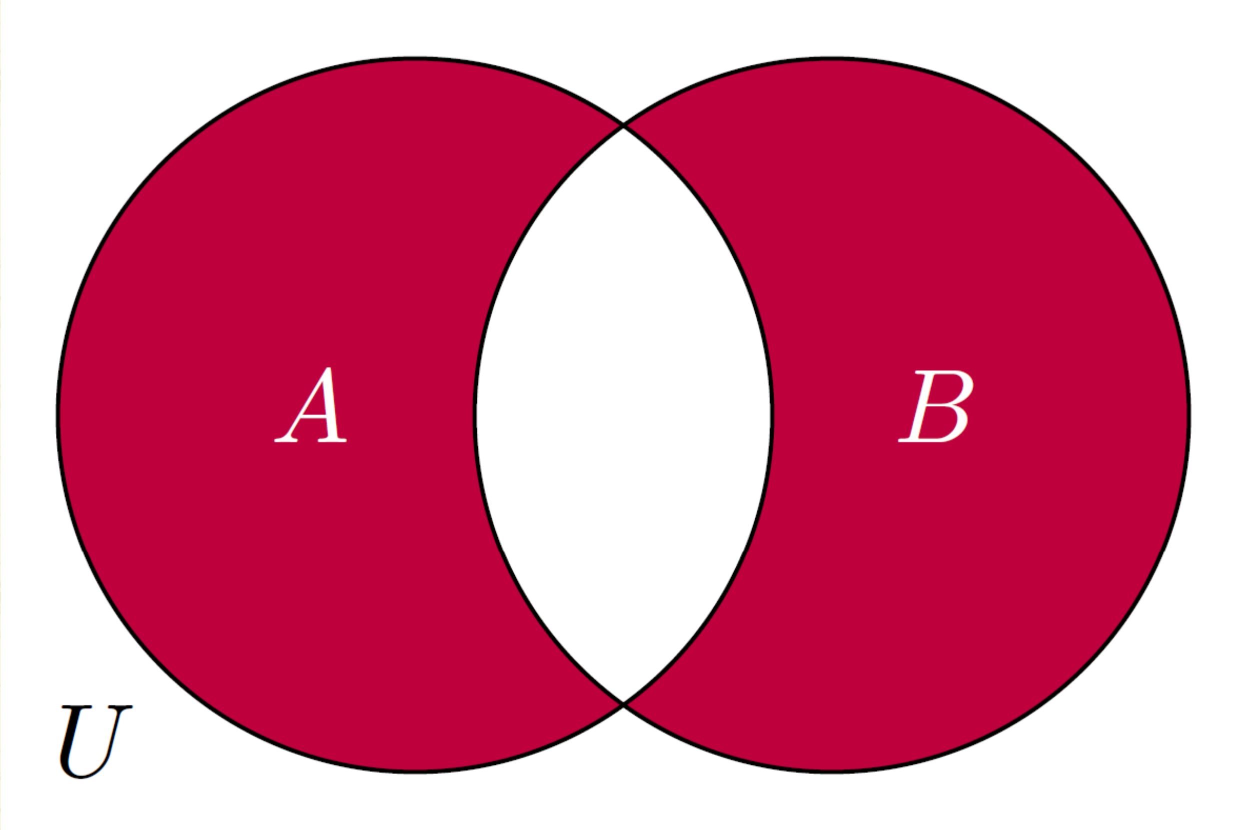
The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={},
overlap=1cm, shade=purple]
\fillNotAorB % The set U \ (A U B) = U - A - B
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Set Operations: Venn Diagrams



The area representing $A \triangle B$ (Symmetric difference of A and B) is filled with wine purple color. Miriam Briskman, CC BY-NC 4.0.

Set Operations: Venn Diagrams

The code that generates the previous PDF diagram:

```
\documentclass{standalone}

% Imported packages:
\usepackage{venndiagram} % Needed for drawing Venn diagrams

\begin{document}
\begin{venndiagram2sets}[showframe=false, hgap=0.2cm, vgap=0.2cm, labelA={}, labelB={},
overlap=1cm, shade=purple]
\fillANotB \fillBNotA % The set (A - B) U (B - A)
% The code below sets the locations of the labels $A$, $B$, and $U$ in the diagram:
% [You don't need to change the code below:]
\setpostvennhook{%
```

Any questions so far?

Union and Intersection of Multiple Sets

Sometimes, we want to find the union or intersection of more than 2 sets. Instead of writing the taunting long notation $A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots \cup A_n$ for the union of all these sets, we can instead write:

$$\bigcup_{1 \le i \le n} A_i = \bigcup A_i$$

(`$\bigcup_{1 \le i \le n} A_i$` and `$\bigcup A_i$`). Similarly, instead of writing $A_1 \cap A_2 \cap A_3 \cap A_4 \cap \cdots \cap A_n$ for the intersection of all these sets, we can instead write:

$$\bigcap_{1 \le i \le n} A_i = \bigcap A_i$$

(`$\bigcap_{1 \le i \le n} A_i$` and `$\bigcap A_i$`). You can replace i with any other variable of your choice.

Set Laws/Identities

The table below contains several **set laws** or **identities**, which are pairs of sets that are equal to each other (we will show that these laws are true in Topic 3:)

Name	**Intersection (∩) version**	**Union (∪) version**
Identity Law	$A \cap U = U \cap A = A$	$A \cup \emptyset = \emptyset \cup A = A$
Null Law	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Idempotent Law	$A \cap A = A$	$A \cup A = A$
Complement Law	$A \cap A^c = \emptyset$	$A \cup A^c = U$
Commutative Law	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative Law	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive Law	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption Law	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
DeMorgan's Law	$(A \cap B)^c = A^c \cup B^c$	$(A \cup B)^c = A^c \cap B^c$
Double Complement (Involution) Law	$(A^c)^c = A^{cc} = A$	

Set Laws/Identities

Any questions so far?

More on Sequences

On slide 6, we learned what sequences are; we promised to resume our discussion on them, which we do now.

Since a sequence is an ordered container of elements, it makes sense to refer to its terms by their position in the sequence. In fact, we can use variables to denote the terms: a_1 is the first (leftmost) term, a_2 is the next term, etc.

Some books begin counting at 0, as in a_0. Also, many programming languages have arrays starting with the index of 0. Also, it is common to use lowercase letters to name terms, e.g., b_{12}, k_3, or x_{109}.

Example: In the sequence $\langle 10, 11, 12, 13, \ldots \rangle$, we denote the terms as $a_1 = 10$, $a_2 = 11$, $a_3 = 12$, $a_4 = 13$, etc. A shorthand for a sequence of n terms is $(a_i)_{i=1}^{n} = (a_1, a_2, \ldots, a_n) = \langle a_1, a_2, \ldots, a_n \rangle$ (`$(a_i)^n_{i=1}$`). The variable n usually stands for the position of the last term in the sequence but can also be replaced with ∞ for infinite sets.

As we've seen in our sequence examples so far, we can list the terms of a sequence (if it is finite and short) and use the ellipsis $(\ldots)$ to indicate continuation. Moreover, sequences with special term patterns have names, such as **arithmetic sequences** and **geometric sequences**; we'll define those in the next slides.

More on Sequences

The **sums**, or **series**, of a sequence is the sum of all or a part of the sequence's terms.

Three **sum notations** for a sequence whose name is A are:

$$\sum_{a \in A} a = \sum_{i=1}^{n} a_i = \sum_{1 \le i \le n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

The first notation is written in LaTeX as `$\sum_{a \in A} a$` ("the sum of all a in A") and the other two as `$\sum_{i=1}^n a_i$` and `$\sum_{1 \le i \le n} a_i$` ("the sum of all a_i such that i goes from 1 to n".)

Note how closely the sum notations resemble `for` loops in a programming language!
Fun class activity: How do you write $\sum_{a \in A} a$ as a `for` loop in Java/C++? What about $\sum_{i=1}^{n} a_i$?

<u>Example</u>: Let $S = \langle 7, 2, 5, 1 \rangle$. Then, $\sum_{k=1}^{4} s_k = 7 + 2 + 5 + 1 = 15$.

More on Sequences

Besides sums, we also have **product notations**:

$$\prod_{a \in A} a = \prod_{i=1}^{n} a_i = \prod_{1 \le i \le n} a_i = a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n.$$

The first notation is written in $\LaTeX$ as `$\prod_{a \in A} a$` ("the product of all a in A") and the other two as `$\prod_{i=1}^n a_i$` and `$\prod_{1 \le i \le n} a_i$` ("the product of all a_i such that i goes from 1 to n".)

Note how the ellipsis $\ldots$ (`$\dots$`) turned into middle dots $\cdots$ (`$\cdots$`) when we include it between two $+$s (additions) or two $\cdot$s (multiplications). That is, `$\dots$` adjusts itself to the expression's context.

Fun class activity: How do you write $\prod_{a \in A} a$ as a `for` loop in Java/C++? What about $\prod_{i=1}^{n} a_i$?

<u>Example</u>: Let $S = \langle 7, 2, 5, 1 \rangle$. Then, $\prod_{k=1}^{4} s_k = 7 \cdot 2 \cdot 5 \cdot 1 = 70$.

More on Sequences

Fun class activity:

1. Let $A = \langle 1, 2, \ldots, 10 \rangle$.
    1. How much is $\sum_{j \in A} j$?
    2. How much is $\prod_{m=1}^{5} a_m$?
2. Let $B = \langle 4, -4, 3, -3, 2, -2, 1, -1, 0 \rangle$.
    1. How much is $\sum_{b \in B} b$?
    2. How much is $\prod_{m=1}^{|B|} p_m$?

We use size ($|B|$) and element inclusion ($j \in A$) notation for sequences, too. However, there is a difference between sets and sequences in terms of size: $|\{1, 1, 1, 1\}| = 1$ because we just repeat the same set element, while $|\langle 1, 1, 1, 1 \rangle| = 4$ since every term has a unique position in the sequence and is, therefore, meaningful (it isn't just repetition.)

More on Sequences

Depending on the sequence pattern, you may sometimes find a handy **formula** for its sums: with such a formula, each time you are given a number n, simply plug the given n into the formula, and you'll quickly find the sum.

Let's consider the sequence $S = \langle 1, 2, 3, \dots, n \rangle$ for some fixed integer n. It turns out that

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The sum above has a special name: an **arithmetic series**, and the underlying sequence S is an **arithmetic sequence**. In this kind of sequence, the difference between each two adjacent terms is the same (in the case of S, it is $3 - 2 = 2 - 1 = 1$.)

How was it discovered? A legend/rumor tells that when Carl Friedrich Gauss, an 18th century mathematician, was in 3rd grade, he quickly computed the sum $1 + 2 + \cdots + 10$ by noticing that $(10 + 1)+(9 + 2)+(8 + 3)+(7 + 4)+(6 + 5) = 11 \cdot 5 = \frac{10 \cdot 11}{2} = 55$. We will prove that the formula is correct later on.

More on Sequences

In a general case, the sum of an arithmetic sequence of the form $A = \langle a_1,\ a_2,\ a_3,\ \ldots,\ a_n\rangle = \langle a_1,\ a_1 + d,\ a_1 + 2d,\ \ldots,\ a_1 + d\cdot(n-1)\rangle$, where d is the common difference, is

$$\sum_{a\in A} a = \frac{n\cdot[2a_1 + d\cdot(n-1)]}{2} = \frac{n(a_1 + a_n)}{2}.$$

Let's get back to $S = \langle 1, 2, 3, \ldots, n\rangle$. If you apply the product notation to S, you get $\boldsymbol{n}$ **factorial**:

$$n! = \prod_{i=1}^{n} i = 1\cdot 2\cdot 3\cdot\cdots\cdot n.$$

Note that $0! = 1$, $1! = 1$, $2! = 2\cdot 1 = 2$, etc. It doesn't have a formula, but most calculators have an $n!$ button.

More on Sequences

A geometric sequence of the form $G = \langle a, a \cdot r, a \cdot r^2, a \cdot r^3, \ldots \rangle$ is a sequence of numbers in which each term after the first is obtained by multiplying the previous term by a fixed common ratio r.

The nth term of a geometric sequence is $a_n = a_1 \cdot r^{n-1}$ (just plug in a_1, r, and n to get the number.)

The sum of a geometric sequence has a formula: if the sequence is finite, like $G = \{1, 2, 4, 8, 16\}$ ($r = 2$), the formula is

$$\sum_{i=1}^{n} a_i = a_1 \cdot \frac{1 - r^n}{1 - r}, \ (r \neq 1).$$

if the given sequence is infinite, and if $|r| < 1$, like $G = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ ($r = \frac{1}{2}$), the formula is

$$\sum_{a \in G} a = \frac{a_1}{1 - r}.$$

More on Sequences

Another special sequence and its sum: $T = \langle 1, 4, 9, \ldots, n^2 \rangle$, the sequence of squares of integers, has the sum

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- A sequence whose terms always grow (e.g., $\langle 1, 4, 9, \ldots, n^2 \rangle$,) always diminish (e.g., $\langle 10^n, \ldots, 10^2, 10^1, 10^0 \rangle$,) or are always constant is called a **monotone sequence**.
- A sequence whose terms are always constant (e.g., $\langle 7, 7, 7, \ldots, 7 \rangle$) is called a **constant sequence**.

Fun class activities: (1) What is the sum $\sum_{i=1}^{15} i$? (2) The sum $\sum_{i=1}^{5} i^2$? (3) $5!$ (4) The sum $\sum_{i=0}^{\infty} \frac{1}{2^i}$?

More on Sequences

Any questions so far?

Sets & Sequences: Sources

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