Logic & Simple Proof Methods

CISC 2210 — CUNY Brooklyn College
Lecture Notes

Logic: Definition

The study of **logic** focuses on the principles of valid reasoning and argumentation. In discrete mathematics, logic is used to analyze and manipulate statements in a formal manner.

Propositional logic is a branch of logic that deals with statements that can be either **true** or **false**, but not both.

A **proposition** is any declarative statement/claim that has a definite truth value (either true or false, but not both.)

For example, the statement "$2+2=4$" is a true proposition. Conversely, the statement "all prime numbers are odd" is a false proposition (*Why is this latter proposition false? **Hint**: can you name a prime number that isn't odd?*). More examples:

- "There are ∞-many integers" - *true*. (Same as claiming: $|\mathbb{Z}| = \infty$ (`$|\mathbb{Z}| = \infty$`).)
- "The number $\sqrt{2}$ is rational" - *false*. (Same as claiming: $\sqrt{2} \in \mathbb{Q}$ (`$\sqrt{2} \in \mathbb{Q}$`).)
- "The number $\sqrt{4}$ is rational" - *true*. (Same as claiming: $\sqrt{4} \in \mathbb{Q}$ (`$\sqrt{4} \in \mathbb{Q}$`).)

Logic: Definition

Understanding logic is essential for computer scientists, even those of you who do not plan to study theory or pure mathematics.

Logical reasoning is the foundation of programming, as it enables developers to construct precise and error-free code.

Boolean logic, derived from propositional logic, is crucial for correctly designing computer circuits and algorithms. In addition, Logical thinking helps in debugging and verifying the correctness of software and systems.

By analyzing statements in terms of logical structure, computer scientists can write more efficient and reliable programs. For example, one may find and replace a long `if` statement with a shorter one that returns the exact same results but runs much faster.

Additionally, logic is fundamental in **artificial intelligence** and **database query languages**, where reasoning and decision-making play a key role.

Propositional Logic and Operators

We can **combine** multiple propositions, which we'll conveniently denote as p, q, r, s, t, etc., using **logical operators**:

Operator	Name	Usage Example	$\LaTeX$	Read as...	True when...
$\neg$	Negation ('Not')	$\neg p$	`$\neg p$`	"Not p"	p is false.
$\wedge$	Conjunction ('And')	$p \wedge q$	`$p \wedge q$`	"p and q"	both p and q are true.
$\vee$	Disjunction ('Or')	$p \vee q$	`$p \vee q$`	"p or q"	p or q or both are true.
$\oplus$	Exclusive Or ('Xor')	$p \oplus q$	`$p \oplus q$`	"p xor q"	p or q are true, but not both.
$\rightarrow$	Implication	$p \rightarrow q$	`$p \rightarrow q$`	"p implies q"	p is false or q is true.
$\leftrightarrow$	biconditional	$p \leftrightarrow q$	`$p \leftrightarrow q$`	"p if and only if q"	p and q are both true or both false.

In some books, implication is denoted as $p \Rightarrow q$ (`$p \Rightarrow q$`) and the biconditional as $p \Leftrightarrow q$ (`$p \Leftrightarrow q$`), but, in our course, we'll use $\Rightarrow$ and $\Leftrightarrow$ later for a different purpose that also related to logic.

Propositional Logic and Operators

As we emphasized previously, a proposition can be either "true" or "false". We can use propositional variables to represent real-life events, such as "it will rain today" or "we will go watch the soccer game today". A complex logic expression will be one that combines one or more simple propositional variables together using the logical operators we've seen on the previous slide.

Example: The proposition:

> It won't rain today, and we will go watch the soccer game today.

contains the two simple events "it will rain today" or "we will go watch the soccer game today". However, we first negated the 1st event (so it turned into "it won't rain today") and then ANDed it with the 2nd event.

If we denote $p = (\text{"it will rain today"})$ and $q = (\text{"we will go watch the soccer game today"})$, then we can represent our proposition with the expression: $(\neg p) \wedge q = \neg p \wedge q$.

Propositional Logic and Operators

Consider a different statement:

> If it rains today, then we will stay home today.

This statement, which is written as "if p then q", can also be equivalently restated as "We will stay home today if it rains today". (In this example, $p = (\text{"it will rain today"})$ and $q = (\text{"we will stay home today"})$.)

Note that a conditional statement of the form "if p then q" is true either if p is false (regardless of what q is,) or if q is true (regardless of what p is.) In other words, the only case when this statement is false is when p is true but q is false (that is, when it indeed rains, but, despite this, we still go outside.)

This description of a conditional / implication statement tells us that we can express the proposition at the top as: $p \to q = (\neg p) \vee q = \neg p \vee q$ (not p, or q).

Propositional Logic and Operators

Here is yet another statement:

> We can drink coffee or tea and not eat any donuts, or just drink coffee.

This statement sounds quite odd and abstract, but we can still express it with our logic notation. If we denote $p = (\text{"drink coffee"})$, $q = (\text{"drink tea"})$, and $r = (\text{"eat donuts"})$, then the statement can be expressed as: $((p \vee q) \wedge (\neg r)) \vee p = (p \vee q) \wedge \neg r \vee p$.

We soon see that this proposition can be re-phrased is a shorter and simpler way. This is true for other long propositions, too: there might be a way of re-writing a proposition such that it uses fewer variables or instances of variables, but still fully retains its truth value.

Quick example: The proposition: "This coffee is great; this coffee is great; this coffee is great!", which we denote as $p \wedge p \wedge p$, can be re-phrased as simply "This coffee is great": p (if we are referring to the same exact coffee sample.)

Propositional Logic and Operators

Fun class activity:

1. Represent the following compound propositions using logical operators. You can use the variables p, q, and r to represent simple statements inside those propositions:
    1. The integer 3 is even, and $2 - 1 = 1$.
    2. The integer 9 is divisible by 3, or the integer 9 is not divisible by 3.
    3. If Earth is flat, then today is a Monday or Wednesday.
    4. The integer n is equal to 2 if and only if the integer n is bigger than 1 and the integer n is smaller than 3.
2. For each of the compound propositions above, indicate if it is true or false. Explain your answer.

Precedence of operations: Parentheses () are evaluated before negation ($\neg$); negation before conjunction ($\wedge$); conjunction before exclusive disjunction ($\oplus$); exclusive disjunction before disjunction ($\vee$); disjunction before implications ($\rightarrow$); and implications before biconditionals ($\leftrightarrow$).

Any questions so far?

Propositional Logic and Operators

Let talk a bit more about the statement $p \to q$ (= "p implies q" = "if p, then q".)

There is a relationship between $p \to q$ and the statements $q \to p$, $\neg p \to \neg q$, and $\neg q \to \neg p$:

The statement:	is called the:
$q \to p$	**converse** of the statement $p \to q$.
$\neg p \to \neg q$	**inverse** of the statement $p \to q$.
$\neg q \to \neg p$	**contrapositive** of the statement $p \to q$.

We will soon see that $p \to q$ is equivalent to its contrapositive statement $\neg q \to \neg p$, but isn't necessarily equal to the converse statement $q \to p$. [Note that the converse: $q \to p$, however, is, equivalent to the inverse: $\neg p \to \neg q$.]

By saying 'equivalent', we mean that the two statements have the same truth value, no matter what the truth values of the individual statements p and q are. In other words, $\neg q \to \neg p$ is just another way to say $p \to q$ and vice versa.

Propositional Logic and Operators

Example: Consider the following implication:

If it is raining, then the ground is wet.

Symbolically, let $p = (\text{"it is raining"})$ and $q = (\text{"the ground is wet"})$. Then, the implication is written as $p \to q$.

The **converse** ($q \to p$) of this statement is:

If the ground is wet, then it is raining.

The converse is not always true. The ground could be wet for other reasons (e.g., someone watered the lawn).

On the next slide, we bring the corresponding examples for the inverse and contrapositive statements.

The **inverse** ($\neg p \rightarrow \neg q$) of $p \rightarrow q$ is:

> If it is not raining, then the ground is not wet.

The inverse, just like the converse, is also not necessarily true. The ground could still be wet from other sources.

Finally, the **contrapositive** ($\neg q \rightarrow \neg p$) of $p \rightarrow q$ is:

> If the ground is not wet, then it is not raining.

The contrapositive is *always logically equivalent* to the original implication $p \rightarrow q$.

BTW, in the implication $p \rightarrow q$, the proposition p is sometimes called the **hypothesis** or the **premise**, while q is called the **conclusion**. Also, p is called a **sufficient condition** for q, and q is called a **necessary condition** for p.

Quantifiers

A **quantifier** is a logical symbol that specifies how many elements of a domain satisfy a given property.

There are two main types of quantifiers in mathematical logic:

- **Universal quantifier** $\forall$ (`$\forall$`), read as "for all".
- **Existential quantifier** $\exists$ (`$\exists$`), read as "there exists".

The **universal quantifier** states that a property holds for all elements in a given set. It is usually used in the following format:

$\forall x \in S, p(x)$ (`$\forall x \in S, p(x)$`)

which means "for all x in the set S, the property $p(x)$ it true." (Here, $p(x)$ is a proposition that uses x in it.)

Quantifiers

Example: Consider the statement:

$$\forall x \in \mathbb{N}, x + 0 = x.$$

This means "for all natural numbers x, adding 0 to x results in the same object x." This is a true statement in standard arithmetic.

The **existential quantifier** states that there is at least one element in the set for which a property holds. Its usual format is:

$$\exists x \in S, p(x)$$ (`$\exists x \in S, p(x)$`)

which means "for at least one x in the set S, the property $p(x)$ is true."

Quantifiers

Example: Consider the statement:

$$\exists m \in \mathbb{Z}, m^2 = 25.$$

This means "there exists an integer m such that $m^2 = 25$."

This is true since $x = 5$ and $x = -5$ satisfy the equation.

Because the statement uses a $\exists$ quantifier, it is actually enough just to point at one such integer, e.g., $x = 5$, to prove the correctness of this statement.

Fun fact: note that $\forall$ is an upside-down letter A (reminiscent of the word 'all') and that $\exists$ is a flipped letter E (reminiscent of the word 'exists').

Quantifiers: How to Prove/Disprove

To prove a universal statement, we must show that $p(x)$ is true for *all* the elements in the set. To disprove a universal statement, on the other hand, it is enough to find a single **counterexample** where $p(x)$ is false.

To prove an existential statement, we only need to find *one* example where $p(x)$ is true. To disprove an existential statement, on the other hand, we must show that $p(x)$ is false for *all* the elements the set.

Quantifier	How to Prove	How to Disprove
Universal ($\forall$)	Show $p(x)$ is true for all x.	Find a single counterexample where $p(x)$ is false.
Existential ($\exists$)	Find at least one x where $p(x)$ is true.	Show $p(x)$ is false for all x in the set.

Bottom line: Generally, universal statements are harder to prove but easier to disprove. Existential statements are easier to prove but harder to disprove.

Quantifiers: How to Prove/Disprove

Fun class activities: For each of the following quantified propositions, determine whether the statement is true or false. If it is true, provide a proof. If it is false, give a counterexample.

1. Consider the set of natural numbers $\mathbb{N}$.
   Determine the validity of the following statement: $\forall x \in \mathbb{N}, x + x \geq x.$
2. Consider the set of integers $\mathbb{Z}$.
   Prove or disprove the following statement: $\exists x \in \mathbb{Z}, 1 < x < 2.$
3. Consider the set of real numbers $\mathbb{R}$.
   Analyze the truth value of the following statement: $\forall x \in \mathbb{R}, x^2 \geq x.$
4. Consider the set of positive integers $\mathbb{P}$.
   Evaluate the truth of the following statement: $\exists x \in \mathbb{P}, x > 100.$

Quantifiers: How to Prove/Disprove

Any questions so far?

Truth Tables

A convenient way of analyzing the truthfulness of a statement is by describing it using a truth table. A **truth table** lists all the possible cases for all the involved variables $p, q, r, \ldots$ in the statement and finds whether the statement as a whole is true or false in each such case.

To get started with truth tables, let's first describe the 6 logical operators that we listed on slide 4, one truth table per operator example. This currect slide shows the first 3 of the operators. In our truth tables, 1 means 'true' and 0 means 'false':

p	$\neg p$
0	1
1	0

Truth table for negation: $\neg p$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Truth table for conjunction: $p \wedge q$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Truth table for disjunction: $p \vee q$

Truth Tables

Here are the truth tables for the remaining 3 operator examples:

p	q	$\boldsymbol{p \oplus q}$
0	0	0
0	1	1
1	0	1
1	1	0

Truth table for exclusive disjunction: $p \oplus q$

p	q	$\neg p$	$\boldsymbol{p \to q\ (= \neg p \vee q)}$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Truth table for implication: $p \to q$

p	q	$\neg p$	$\neg q$	$p \to q\ (= \neg p \vee q)$	$q \to p\ (= \neg q \vee p)$	$\boldsymbol{p \leftrightarrow q\ (= (p \to q) \wedge (q \to p))}$
0	0	1	1	1	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	0
1	1	0	0	1	1	1

Truth table for the biconditional: $p \leftrightarrow q$

Truth Tables

Now that we know the behavior of each of the 6 logical operators, we can build truth tables for any statement / expression as wanted. Let's for example, build the truth table for the expression $(p \vee q) \wedge \neg r \vee p$ that we got on slide 7:

p	q	r	$p \vee q$	$\neg r$	$(p \vee q) \wedge \neg r$	$\boldsymbol{(p \vee q) \wedge \neg r \vee p}$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	1	1	1
1	0	1	1	0	0	1
1	1	0	1	1	1	1
1	1	1	1	0	0	1

Truth table for $(p \vee q) \wedge \neg r \vee p$

Truth Tables

Suppose you are given two logical expressions, e.g., $p \to q$ and $\neg q \to \neg p$, and want to check if they are equivalent or not. You can do so by constructing the truth tables for each of the 2 expressions:

p	q	$\neg p$	$\boldsymbol{\neg p \vee q}$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

p	q	p'	q'	$\boldsymbol{\neg\neg q \vee \neg p = q \vee \neg p}$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	0	0	1

Because the sequences of truth values that we got under the $\neg p \vee q$ and $\neg\neg q \vee \neg p = q \vee \neg p$ columns are exactly the same: $\langle 1, 1, 0, 1 \rangle$, this means that the statement are equivalent, so $p \to q = \neg q \to \neg p$ is a valid equivalence!

The next 2 slides introduce additional such **equivalence**; we use $\Leftrightarrow$ to indicate that the two expressions are equivalent.

Equivalences, Part 1

Name	Conjunction ($\wedge$) version	Disjunction ($\vee$) version
Identity Law	$p \wedge \mathbf{1} \Leftrightarrow p$	$p \vee \mathbf{0} \Leftrightarrow p$
Null Law	$p \wedge \mathbf{0} \Leftrightarrow \mathbf{0}$	$p \vee \mathbf{1} \Leftrightarrow \mathbf{1}$
Idempotent Law	$p \wedge p \Leftrightarrow p$	$p \vee p \Leftrightarrow p$
Complement Law	$p \wedge \neg p \Leftrightarrow \mathbf{0}$	$p \vee \neg p \Leftrightarrow \mathbf{1}$
Commutative Law	$p \wedge q \Leftrightarrow q \wedge p$	$p \vee q \Leftrightarrow q \vee p$
Associative Law	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$(p \vee q) \vee r \Leftrightarrow p \vee (q \wedge r)$
Distributive Law	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
Absorption Law	$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$
DeMorgan's Law	$p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q)$	$p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$
Double Negation Law	$\neg(\neg p) \Leftrightarrow \neg\neg p \Leftrightarrow p$	

Equivalences, Part 2

Name	Equivalence
Contrapositive Law	$p \to q \Leftrightarrow \neg q \to \neg p$
Implication Law #1	$p \to q \Leftrightarrow \neg p \vee q$
Implication Law #2	$p \to q \Leftrightarrow \neg(p \wedge \neg q)$
Implication Law #3	$p \vee q \Leftrightarrow \neg p \to q$
Implication Law #4	$p \wedge q \Leftrightarrow \neg(p \to \neg q)$
Conjunction of Implications Law #1	$(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$
Conjunction of Implications Law #2	$(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)$
Biconditional Law	$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
Exportation Law	$(p \wedge q) \to r \Leftrightarrow p \to (q \to r)$
Reduction to Absurdity Law	$p \to q \Leftrightarrow (p \wedge \neg q) \to 0$

Equivalences

Some rules (and tips) on building truth tables for logic expressions:

- The few leftmost columns of your table should be the simple variables the expression contains. For example, if your expression has the variables p, q, and r in it, your table should start with 3 columns, one for each of p, q, and r.
- The number of rows that your table will have can be calculated as follows: $2^{(\text{amount of variables})}$. For example, if your expression has the variables p, q, and r in it, the truth table will feature $2^3 = 8$ rows of bits.
- To make the evaluation of expressions gradual and easy, include additional columns in your table that correspond to intermediate expressions that build up the final expression. For example, if your expression is $\neg(p \vee q)$, don't include just the columns p, q, and then right away $\neg(p \vee q)$. Instead, include also a column for the $\vee$ operation: $p \vee q$, which is an intermediate operation that you perform before applying the negation operation.
- The 'variables' 1 and 0 are special: while variables like p and q vary (each of them can be either 0 or 1,) 1 and 0 are actually constants. If you need to include a column for 1, all the bits across this column will be 1s. Similarly, if you need to include a column in the truth table for 0, all the bits across this column will be 0s.

Equivalences

- If you need to check whether two expressions are equivalent, you'll need to build 2 tables, one for each expression. You may, if wanted, 'glue' these tables to one another to create a single, longer table.
- Two expressions are equivalent if and only if the sequences of digits underneath the columns for each of these expressions are identical. In the example for $p \to q$ and $\neg q \to \neg p$, the sequence of digits for each of these was $(1, 1, 0, 1)$.
- If the sequences of digits differ from one another, even in just one of the digits, the two expressions are not equivalent.

Let's verify a couple more identities. The truth table for the equivalence $p \wedge 1 \Leftrightarrow p$ is:

1	$\boldsymbol{p}$	$\boldsymbol{p \wedge 1}$
1	0	0
1	1	1

Since the digits under p and under $p \wedge 1$ are the same (i.e., $(0, 1)$), we successfully confirmed the equivalence.

Equivalences

The truth tables for the equivalence $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ are:

p	q	r	$q \wedge r$	$\boldsymbol{p \vee (q \wedge r)}$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

p	q	r	$p \vee q$	$p \vee r$	$\boldsymbol{(p \vee q) \wedge (p \vee r)}$
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Because the sequences of bits that we got under the $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ columns are exactly the same: $(0, 0, 0, 1, 1, 1, 1, 1)$, this means that the expressions are equivalent!

Equivalences

Let's check if the expressions: $p \vee q \wedge r$ and $q \vee p \wedge r$ are equivalent or on:

p	q	r	$q \wedge r$	$\boldsymbol{p \vee q \wedge r}$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

p	q	r	$p \wedge r$	$\boldsymbol{q \vee p \wedge r}$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	1
0	1	1	0	1
1	0	0	0	0
1	0	1	1	1
1	1	0	0	1
1	1	1	1	1

Because the sequence of digits that we got under the $p \vee q \wedge r$ column is different from the one under the $q \vee p \wedge r$ column, these two expressions aren't equivalent. We say: $p \vee q \wedge r \not\Leftrightarrow q \vee p \wedge r$ (`$\not\Leftrightarrow$`).

Equivalences

We call the number $\mathbf{1}$ and each one of the equivalences we've seen a **tautology** (= a statement that is always true). On the other hand, the false statement $\mathbf{0}$ and any other false statement like $p \Leftrightarrow \neg p$ is called a **contradiction**.

Fun in-class exercises:

1. Prove the following equivalences by building truth tables for them:
   a. $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r$
   b. $p \vee p \wedge q \Leftrightarrow p$
2. Check if the following pairs of expressions are equivalent or not by building truth tables for them:
   a. $\neg(p \vee q)$ and $\neg p \vee \neg q$
   b. $p \wedge \neg p \vee \neg q$ and $p \wedge \neg q \vee \neg q$

Equivalences

Any questions so far?

Simple Proof Methods

Why do you, as computer science majors, even need to study about proofs?

Understanding and writing mathematical proofs is essential for computer science students because proofs develop rigorous logical reasoning skills, which are fundamental to programming and algorithm design.

For example, proving that an algorithm always terminates and produces the correct result ensures its correctness before implementation.

Proof techniques such as induction, which we will study later this course, are especially useful in analyzing recursive functions and proving properties of data structures like trees and graphs, which we will cover later, too.

By learning to read and construct proofs, you enhance your ability to write clear, efficient, and bug-free code and show the fact that your code is correct to your fellow developers.

Simple Proof Methods

Before diving into proofs, let's first define some key terms in this area of math:

A **definition** provides a precise meaning or notation for a mathematical concept. It establishes the criteria that an object (e.g., a number) must satisfy to belong to a certain category. We can state or establish definitions; we don't prove them.

Examples:

Definition. [Even integers] An integer n is called **even** if there exists an integer k such that $n = 2k$.

Definition. [Odd integers] An integer n is called **odd** if there exists an integer k such that $n = 2k + 1$.

In $\LaTeX$, definitions are written inside the `definition` environment; see slide 36 for the full $\LaTeX$ code.

Simple Proof Methods

A **theorem** is a mathematical statement that has already been proven or can be proven to be true based on definitions or previously proved theorems.

Examples:

Theorem. [Sum of even integers] If m and n are both even integers, then their sum, $m+n$, is also even.

Theorem. [Infinity of $\mathbb{Z}$] There exist infinitely-many integers. That is, $|\mathbb{Z}| = \infty$.

Theorem. [Solutions of a quadratic equation] There exist at most 2 real solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

In $\LaTeX$, theorems are written inside the `theorem` environment; see slide 36 for the full $\LaTeX$ code.

Simple Proof Methods

A **proof** is a logical argument that establishes the truth of a theorem using deductive reasoning. Proofs don't need to be too long, technical, or elaborate, but they do need to be clear and unambiguous.

Examples:

Proof. [Sum of even integers] Let m and n be two even integers. According to the definition of even integers, there exists $k_1 \in \mathbb{Z}$ such that $m = 2k_1$ and $k_2 \in \mathbb{Z}$ such that $n = 2k_2$.
Notice that

$$m + n = 2k_1 + 2k_2 = 2(k_1 + k_2),$$

which is, by definition, an even integer because $(k_1 + k_2)$ is an integer. ■

In $\LaTeX$, proofs are written inside the `proof` environment; see slide 36 for the full $\LaTeX$ code.

Simple Proof Methods

A **corollary** is a result that follows directly from a theorem with little or no additional proof.

Example:

Corollary. [Square of an odd integer] If n is an odd integer, then n^2 is also odd.

Proof. Let n be an odd integer. According to the definition of odd integers, there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. Notice that

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is, by the definition of odd integers, odd because $(2k^2 + 2k)$ is an integer. ■

In $\LaTeX$, corollaries are written inside the `corollary` environment; see slide 36 for the full $\LaTeX$ code.

Simple Proof Methods

The following $\LaTeX$ source code shows how to create definitions, theorems, proofs, and corollaries:

```
\documentclass[12pt]{article}

\usepackage{amsmath}
\usepackage{amssymb}
\usepackage{amsthm} % The Theorem library.

\setlength{\parskip}{10pt} % Space between paragraphs.
\setlength\parindent{0pt} % Eliminating paragraph indentation.

% Definitions of the definition, theorem, and corollary environments:
\newtheorem{definition}{Definition}
\newtheorem{theorem}{Theorem}
```

Simple Proof Methods: Proofs

proofs.pdf 1 / 1 111%

1

Definition 1. [Even integers] *An integer n is called* **even** *if there exists an integer k such that $n = 2k$.*

Theorem 1. [Sum of even integers] *If m and n are both even integers, then their sum, $m + n$, is also even.*

Proof. [Sum of even integers] Let m and n be two even integers. According to the definition of even integers, there exists $k_1 \in \mathbb{Z}$ such that $m = 2k_1$ and $k_2 \in \mathbb{Z}$ such that $n = 2k_2$.

Notice that

$$m + n = 2k_1 + 2k_2 = 2(k_1 + k_2),$$

which is, by definition, an even integer because $(k_1 + k_2)$ is an integer. □

Corollary 1. [Square of an odd integer] *If n is an odd integer, then n^2 is also odd.*

This is a corollary of a probably more general theorem:

Theorem 2. [Product of odd integers] *If $n_1, n_2, n_3, \ldots, n_m$ are odd integers, then their product, $n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_m$, is also odd.*

Simple Proof Methods

Fun class activities:

1. Prove that the integer 10 is even.
2. Prove that the integer 99 is odd.
3. Prove the following theorem:
   Theorem. [Product of even integers] The product of even integers $n_1, n_2, n_3, \ldots, n_m$ is also even.
4. Prove the following corollary:
   Corollary. [Difference of even integers] If m and n are both even integers, then their difference, $m - n$, is also even.
5. Prove or disprove: If m and n are both odd integers, then their sum, $m + n$, is also odd.
6. **Definition.** [Consecutive integers] Two integers m and n are called **consecutive** if $m = n + 1$ or if $n = m + 1$.
   Prove or disprove: The product of two consecutive integers is even.

Any questions so far?

Simple Proof Methods

All the proofs we've seen so far and wrote so far (as part of the previous class activity) belong to the category of **direct proofs**. In this type of proofs, we are usually told to show that some statement of the form $p \to q$ (or $q \text{ given } p$, $p \text{ implies } q$, etc.,) is true by first assuming that p is true, and then, from there, showing how q is true.

For example, to prove the statement

> If n is even and $k \in \mathbb{Z}$, then kn is also even.

which is of the form $p \to q$, we first assume that n is even and that k is an integer (= the p part, also called the **antecedent**: "preceding in time or order"), and, from these premises, show that kn is even (= the q part, also called the **consequent**: "following in time or order").

Although direct proofs seem to be the most intuitive and direct approach, they are not, however, applicable for all situations. Specifically, for proving certain statements, direct proofs turns out to be quite lengthy, and sometime, *impossible* to write. As such, we resort to a few other proof methods, depending on the given statement to prove.

Simple Proof Methods

Before jumping into a few more involved proof methods, let's first see a couple situations where the proof is easier than we've imagined.

A **trivial proof** is a proof in which the consequent is always true, regardless of the antecedent. This occurs when the statement to be proven holds in all cases.

Example: Prove that for any even integer n, we have $n^2 + 1 > 0$.

Proof. The consequent $n^2 + 1 > 0$ is *always* true for any integer n because squaring any integer results in a non-negative number, and then adding 1 ensures the result is always positive. Since the conclusion is true regardless of whether n is even, the statement is **trivially true** (so the proof is called **trivial**.) ■

Simple Proof Methods

Here is another type of a relatively easy proof method.

A **vacuous proof** is a proof in which the antecedent is always false. Since an implication "if p, then q" is always true when p is false, the proof holds without needing to verify the truthfulness of q.

Example: Prove that for any integer n, if n is both even and odd, then $n^3 - 5n = 42$.

Proof. The antecedent "n is both even and odd" is false because no integer can be both even and odd. Since an implication with a false antecedent is always true, the statement is **vacuously true**. ■

Fun class activities:

1. Prove that if $x = 1$ is an even integer, then $x = 0$.
2. Prove that if $n \in \mathbb{N}$, then $2 + 10 = 12$.

Simple Proof Methods

The concept of **divisibility** is fundamental in number theory and discrete mathematics. We already, in fact, talked about divisibility when we mentioned even and odd integers which are divisible (or, not divisible) by 2.

Definition. [Divisibility] An integer m is **divisible** by an integer $n \neq 0$ if there exists a $k \in \mathbb{Z}$ such that $m = k \cdot n$.

We denote this by $n \mid m$ (`$n \mid m$`), which reads as "n divides m", "m is divisible by n", or "m is a multiple of n".

For example, 12 is divisible by 3 because there exists a $k \in \mathbb{Z}$, specifically $4 \in \mathbb{Z}$, such that $12 = 4 \cdot 3$, so we write $3 \mid 12$.

On the other hand, 10 isn't divisible by 3 because there exists no $k \in \mathbb{Z}$ for which $10 = 3k$. This is because $k = \frac{10}{3} = 3.\overline{3}$, which isn't an integer. As such, to show an indivisibility, we say $3 \nmid 10$ (`$3 \not\mid 10$`).

[BTW, the bar over the digit 3 after the decimal point in the number $3.\overline{3}$ (`$3.\overline{3}$`) means that 3 keeps repeating infinitely after the decimal point: $3.33333\ldots$.]

Simple Proof Methods

Certain numbers are divisible by many integerts, while others are divisible by only a couple positive integers. These latter numbers have a special name.

Definition. [Prime numbers] A **prime number** $p > 1$ is an integer that has exactly two positive divisors: 1 and p (itself).

Formally, an integer p is prime if, for all positive integers d, the only solutions to $d \mid p$ are $d = 1$ and $d = p$.

For example, $2, 3, 5, 7, 11, \ldots$ are all prime numbers.

Numbers that have more than two divisors are called **composite numbers**, such as $4, 6, 8, 9, 10, \ldots$.

Bonus question: if p and q are both primes, when is $p + q$ an **odd** integer?

Simple Proof Methods

Definition. [Absolute values] The **absolute value** of a real number x, denoted $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

($$|x| = \begin{cases}x & \text{ if } x \ge 0, \\-x & \text{ if } x < 0.\end{cases}$$)

Geometrically, the absolute value represents the **distance** of a number from zero on the number line.

For example, $|3| = 3$ and $|-8| = 8$.

Since the absolute value is a **piecewise defined function** (specifically, it has 2 pieces,) this means that proofs concerning absolute values will sometimes need to be done in at least two pieces. A proof of this kind is called a **proof by cases**.

Simple Proof Methods

Consider the following proof by cases:

Theorem. For any real number x, it holds that $|x|^2 = x^2$.

Proof. By the definition of absolute value, we consider two cases:

- **Case 1**: If $x \geq 0$, then $|x| = x$, so: $|x|^2 = x^2$.
- **Case 2**: If $x < 0$, then $|x| = -x$, so: $|x|^2 = (-x)^2 = x^2$.

In both cases, we obtain $|x|^2 = x^2$, proving the theorem. ■

The reason that we needed to look into these 2 separate cases is that the absolute value behaves differently for different values of x, so we needed to verify the equation for all cases to make sure that the equation is true for *any* integer x.

Simple Proof Methods

Fun class activities:

1. Prove that if the integers a and b are each divisible by d, then $a + b$ is also divisible by d.
2. Prove that if m is a multiple of 4, then m^2 is a multiple of 16.
3. Prove that if m is a multiple of 4, then m is a multiple of 2, too.
4. Prove that $n^4 - n^2$ is even for all $n \in \mathbb{N}$ (***hint***: use a proof by cases: (1) n is even, and (2) n is odd.)
5. Prove the **triangle inequality**: $|x + y| \leq |x| + |y|$ for any $x, y \in \mathbb{R}$ (***hint***: use a proof by cases.)
6. Prove that $|xy| = |x| \cdot |y|$ for any $x, y \in \mathbb{R}$ (***hint***: use a proof by cases.)
7. Prove that $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$ (***hint***: use a proof by cases.)

We have two more proof methods to cover before finishing Topic 3: proof by contradiction and proof by contrapositive.

Any questions so far?

Simple Proof Methods

On slide 56 of Topic 2, we presented a table with 19 set identities (= equivalences). It's time to show how to prove some! But first, here's a definition which we already mentioned briefly in Topic 2:

Definition. [Set equality] Two sets S_1 and S_2 are called **equal** if $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. We write: $S_1 = S_2$.

Each of the set identities is provable, so we can label them as theorems. Let's do it for the following identities:

1. $A \cap U = A$ (One of the Identity Laws)
2. $A \cap \emptyset = \emptyset$ (One of the Null Laws)
3. $A \cup B = B \cup A$ (One of the Commutative Laws)
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (One of the Distributive Law)
5. $A \cup (A \cap B) = A$ (One of the Absorption Law)

Simple Proof Methods

1. **Theorem.** For any set A and a universe U in which $A \subseteq U$, the following holds:

$$A \cap U = A.$$

Proof. Let A be any set and U be a universe that contains A. To show that $A \cap U = A$, we need to show that $A \cap U \subseteq A$ and that $A \subseteq A \cap U$:

- [Showing that $A \cap U \subseteq A$] Let x be any element in the set $A \cap U$. This means that x is in the intersection of A and U. This means that $x \in A$ and that $x \in U$. Since, as we just said, $x \in A$, it means that $A \cap U \subseteq A$.
- [Showing that $A \subseteq A \cap U$] Let x be any element in the set A. Because $A \subseteq U$, this means that $x \in U$, too. Since $x \in A$ and $x \in U$, it means that x is in the intersection of A and U, so $x \in A \cap U$, which shows that $A \subseteq A \cap U$.

Because we showed that $A \cap U \subseteq A$ and $A \subseteq A \cap U$, we conclude that $A \cap U = A$ by the definition of set equality. ■

Simple Proof Methods

2. **Theorem.** For any set A, the following holds:

$$A \cap \emptyset = \emptyset.$$

Proof. Let A be any set. To show that $A \cap \emptyset = \emptyset$, we need to show that $A \cap \emptyset \subseteq \emptyset$ and that $\emptyset \subseteq A \cap \emptyset$:

- [Showing that $A \cap \emptyset \subseteq \emptyset$] Let x be any element in the set $A \cap \emptyset$. This means that x is in the intersection of A and $\emptyset$. This means that $x \in A$ and that $x \in \emptyset$. However, because $\emptyset$ is the empty set and doesn't contain any elements in it, our assumption that $x \in A \cap \emptyset$ is false; specifically, $A \cap \emptyset$ doesn't contain any elements in it. But we know that the set that contains no elements is called $\emptyset$, so $A \cap \emptyset \subseteq \emptyset$.
- [Showing that $\emptyset \subseteq A \cap \emptyset$] Consider the set $\emptyset$, which doesn't contain any elements in it. The set $A \cap \emptyset$ contains all the elements of A and of $\emptyset$ in it, which, because the set $\emptyset$ is empty, also contains no elements in it. As such, $\emptyset \subseteq A \cap \emptyset$.

Because we showed that $A \cap \emptyset \subseteq \emptyset$ and $\emptyset \subseteq A \cap \emptyset$, we conclude that $A \cap \emptyset = \emptyset$ by the definition of set equality. ■

Simple Proof Methods

3. **Theorem.** For any sets A and B, the following holds:

$$A \cup B = B \cup A.$$

Proof. Let A and B be any sets. To show that $A \cup B = B \cup A$, we need to show that $A \cup B \subseteq B \cup A$ and that $B \cup A \subseteq A \cup B$:

- [Showing that $A \cup B \subseteq B \cup A$] Let x be any element in the set $A \cup B$. This means that x is in the union of A and B. This means that $x \in A$ or that $x \in B$. In other words, it means that $x \in B$ or that $x \in A$. This means that x is in the union of B and A, so $x \in B \cup A$. This implies $A \cup B \subseteq B \cup A$.
- [Showing that $B \cup A \subseteq A \cup B$] Let x be any element in the set $B \cup A$. This means that x is in the union of B and A. This means that $x \in B$ or that $x \in A$. In other words, it means that $x \in A$ or that $x \in B$. This means that x is in the union of A and B, so $x \in A \cup B$. This implies $B \cup A \subseteq A \cup B$.

Therefore, we conclude that $A \cup B = B \cup A$ by the definition of set equality. ■

Simple Proof Methods

4. **Theorem.** For any sets A, B, and C, the following holds:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof. Let A, B, and C be any sets. To show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, we need to show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

- [Showing that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$] Let x be any element in the set $A \cup (B \cap C)$. This means that x is in the union of A and $B \cap C$. This means that $x \in A$ or that $x \in B$ and $x \in C$. In other words, it means that $x \in A$ or $x \in B$, and that $x \in A$ or $x \in C$. This means that x is in the intersection of $A \cup B$ and $A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. This implies $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
- [Showing that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$] Let x be any element in the set $(A \cup B) \cap (A \cup C)$. This means that x is in the intersection of $A \cup B$ and $A \cup C$. This means that $x \in A$ or $x \in B$, and that $x \in A$ or $x \in C$. In other words, it means that $x \in A$ or that $x \in B$ and $x \in C$. This means that x is in the union of A and $B \cap C$, so $x \in A \cup (B \cap C)$. This implies $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Therefore, we conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by the definition of set equality. ■

Simple Proof Methods

5. **Theorem.** For any sets A and B, the following holds:

$$A \cup (A \cap B) = A.$$

Proof. Let A and B be any sets. To show that $A \cup (A \cap B) = A$, we need to show that $A \cup (A \cap B) \subseteq A$ and that $A \subseteq A \cup (A \cap B)$:

- [Showing that $A \cup (A \cap B) \subseteq A$] Let x be any element in the set $A \cup (A \cap B)$. This means that x is in the union of A and $A \cap B$. This means that $x \in A$ or that $x \in A$ and $x \in B$. In other words, it means that $x \in A$ or $x \in A$, and $x \in A$ or $x \in B$, which is the same as saying $x \in A$ and $x \in A \cup B$. Since $x \in A$, this implies $A \cup (A \cap B) \subseteq A$.
- [Showing that $A \subseteq A \cup (A \cap B)$] Let x be any element in the set A. This means that x is in the union of A with any other set (say, B,) so $x \in A \cup B$. This means that $x \in A$ and $x \in A \cup B$. In other words, it means that $x \in A$ or $x \in A$, and $x \in A$ or $x \in B$. This means that $x \in A$ or that $x \in A$ and $x \in B$, so x is in the union of A and $A \cap B$, so $x \in A \cup (A \cap B)$. This implies $A \subseteq A \cup (A \cap B)$.

Therefore, we conclude that $A \cup (A \cap B) = A$ by the definition of set equality. ■

Simple Proof Methods

Fun class activities: Prove the following identities:

1. $A \cap A = A$ (Idempotent Law)
2. $A \cup A^c = U$ (Complement Law)

Sometimes using a direct proof of a proof by cases isn't easy or isn't short and neat. For example, how do we go about proving that there exist infinitely many integers? A direct proof would most likely require us to list all the integers, but there are infinitely many of them! So what should we do?

If a direct proof doesn't seem appropriate, we'll resort to a **proof by contradiction**: this kind of proof assumes that a given statement (e.g., "there exist infinitely many integers") is false and then derives a contradiction, proving that the assumption we made must have been incorrect. That is, the original statement is, therefore, true.

Simple Proof Methods

It's time to show that there exist infinitely many integers by using a proof by contradiction!

Theorem. [Infinity of $\mathbb{Z}$] There exist infinitely-many integers. That is, $|\mathbb{Z}| = \infty$.

Proof. To obtain a contradiction, assume that there exists a finite number of integers. Since all those integers have different values, one of these integers must be the largest one; let's call it n.
But hey: note that $n + 1$ is an integer. Moreover, note that $n + 1 > n$, which means that $n + 1$ is larger than n.
This is a contradiction because we said that n was the largest of those finitely many integers, which means that $n + 1$ is a brand new integer outside of the finite integers that we assumed. In other words, no matter which finite list of integers you pick, there's always going to be a bigger integer.

This shows that our assumption that there exists a finite number of integers is wrong, so it must be true that there exist infinitely-many integers. ■

Simple Proof Methods

Let's show the validity of another theorem using a proof by contradiction again.

Theorem. [Prime divisibility] A prime number isn't divisible by any other prime number.

Proof. Let $p \in \mathbb{P}$ be a prime number. To obtain a contradiction, assume that p is divisible by some prime q that isn't equal to p. Since q is a prime, it means that $q > 1$ (because 1 isn't a prime number, and prime numbers are always positive integers [other than 1]).

This is a contradiction to the fact that p is a prime, because, according to the definition of prime numbers on slide 44, the only numbers that divide p are 1 and p, so it's false that q, where $q \neq p$ and $q \neq 1$, divides p.

We, therefore, conclude that a prime number isn't divisible by any other prime number. ■

Simple Proof Methods

Next, let's define what irrational number are, and use a proof by contradiction on them.

Definition. [Irrationality] An **irrational** number is one that can't be written as $\frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$ (m and n don't have common factors: $\frac{m}{n}$ is the most simplified fraction.)

Theorem. The number $\sqrt{2}$ is an irrational number.

Proof. To obtain a contradiction, assume that $\sqrt{2}$ is a rational number. This means that $\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ and $n \neq 0$. When squaring both side of the equation, we get: $2 = \frac{m^2}{n^2}$. Multiplying both side by n^2 gives us $2 \cdot n^2 = m^2$. By the definition of even numbers, m^2 is even, and, therefore, m is even. Moreover, if we substitute $m = 2k$, we get $2 \cdot n^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$, so $n^2 = 2k^2$, so n^2, and therefore n, is even. Our conclusion that m is even and n is even is a contradiction to the fact that m and n don't share any common factors.
This means that our assumption that $\sqrt{2}$ is rational is false, so we conclude that $\sqrt{2}$ is an irrational number. ■

Any questions so far?

Simple Proof Methods

Another proof method to try out if a direct proof isn't the best recourse is a **proof by contrapositive**.

If the statement you are trying to prove is of the form of an implication $p \to q$, instead of proving that p implies q (the direct way,) you can prove that $\neg q$ implies $\neg p$. That is, you will prove that $\neg q \to \neg p$, the contrapositive implication, is true.

This proof method is valid since we showed earlier that $p \to q$ is equivalent to $\neg q \to \neg p$.

Theorem. If n^2 is even, then n is even, too.

Proof. We will prove the contrapositive of this statement, i.e., "if n is not even, then n^2 is not even".

Note that saying "not even" is the same as saying "odd", so the contrapositive can be re-written as "if n is odd, then n^2 is odd". We already proved this very statement! See the proof on slide 35. This means that "if n^2 is even, then n is even" is true. ■

Another example:

Theorem. If n^2 is divisible by 4, then n is even.

Proof. We will prove the contrapositive of this statement, i.e., "if n is odd, then n^2 is not divisible by 4".

Let n be indeed an odd integer. By the definition of odd integers, we have $n = 2k + 1$ for some $k \in \mathbb{Z}$. We have:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1,$$

which isn't divisible by 4, as we wished to prove, because $4(k^2 + k) + 1$ leaves a remainder of 1 when divided by 4.

Hence, the statement "if n^2 is divisible by 4, then n is even" is true. ■

Simple Proof Methods

Let's prove the truthfulness of a biconditional, which will use a mix of proof methods:

Theorem. [Exercise 4 on slide 8] Consider an $n \in \mathbb{Z}$. It holds that $n = 2$ if and only if $1 < n < 3$.

Proof. Because of the words "if and only if" (you may sometimes see "iff" instead of "if and only if",) we are working with a biconditional of the form $p \leftrightarrow q$. To show it's true, we need to show both that $p \rightarrow q$ and that $q \rightarrow p$:

- [Showing that if $n = 2$, then $1 < n < 3$ using a direct proof] Assume that $n = 2$. This means that $n = 2 > 1$ and that $n = 2 < 3$. In other words, $1 < n < 3$.
- [Showing that if $1 < n < 3$, then $n = 2$ using a proof by contrapositive] Suppose that $n \in \mathbb{Z}$ and $n \neq 2$. It means that n is any integer besides 2, which, in other words, is either smaller than or equal to 1 or larger than or equal to 3. This means $n \leq 1$ or $n \geq 3$.

because we showed that both implication directions are true, it means that the theorem is true. ■

Simple Proof Methods

Fun class activities: Prove the following:

1. There exist infinitely-many even integers.
2. The number $\sqrt{5}$ is an irrational number.
3. If x^2 is irrational, then x is irrational.
4. If n^3 is an odd integer, then n is odd.
5. Consider an $n \in \mathbb{Z}$. It holds that $n = k$ if and only if $k - 1 < n < k + 1$.
6. No odd integer can be expressed as the sum of 2 even integers.

Simple Proof Methods: Tips

The following tips, based on *Mathematical proofs: A transition to advanced mathematics*, will make your proofs clear and easy to read:

- Don't start sentences or proof with a symbol. Example: Instead of writing:

  $n = 2$ is even.

  write:

  The number $n = 2$ is even.

- Separate symbols with words, unless you list them in a comma-separated list. Example: Instead of writing:

  While $x \in A$, B is an empty set.

  write:

  While $x \in A$, the set B is an empty set.

Simple Proof Methods: Tips

- Unless you answer a question on logic, don't use the symbols $\rightarrow$, $\forall$, $\exists$. Example: Instead of writing:

  The prime p isn't divisible by q, $\forall 1 < q < p$.

  write:

  The prime p isn't divisible by any interger q besides 1 and p.

- Don't mix symbols and text improperly. Example: Instead of writing:

  Some of the integers ≤ 10 are prime.

  write:

  Some of the integers $n \leq 10$ are prime.

- Use a symbol in a theorem only when needed (e.g., when used for an equation inside the theorem.) Example: Instead of writing:

  The square of any even integer m is even.

  write:

  The square of any even integer is even.

Simple Proof Methods: Tips

- If you use a new symbol or notation of your own, always explain what it stands for. Example: Instead of writing:

  Suppose that $n \in \mathbb{Z}^+$.

  write:

  Let $\mathbb{Z}^+$ (`$\mathbb{Z^+}$`) denote the set of positive integers, and suppose that $n \in \mathbb{Z}^+$.

- Use symbols from fixed categories properly. The lowercase letters a, b, c, d, e are used for constants; the letters f, g, h are used for function names; the letters i, j, k, l, m, n, o are used for integers; the letters p, q, r, s, t are used for rational numbers or propositions; the letters u, v, w, x, y, z are used for real numbers; uppercase letters A, B, C, are used for containers such as sets, sequences, etc.
  Example: Instead of writing:

  We have $f \in a$.

  write:

  We have $x \in A$.

Any questions?

Logic & Simple Proof Methods: Sources

Aspnes, James. (2022). Chapter 2. *Notes on Discrete Mathematics*. License: CC BY-SA: Attribution-ShareAlike.

"CS202: Discrete Structures | Units 3 and 4." *Saylor Academy*. License: CC BY: Attribution.

Cusick, Larry. How To Write Proofs.

Davis, Stephen. (2019). Chapters 8 and 9. *A Cool Brisk Walk Through Discrete Mathematics*, version 2.2. License: CC BY-SA: Attribution-ShareAlike.

Doerr, Al and Levasseur, Ken. (2024). Chapter 3. *Applied Discrete Structures*, https://discretemath.org/. License: CC BY-NC-SA: Attribution-NonCommercial-ShareAlike.

Jamaloodee, Mohamed, et al. Chapter 4. *Discrete Math*. Georgia Gwinnett College. License: CC BY-NC: Attribution-NonCommercial.

Levin, Oscar. (2023). Chapters 0.2 and 3. *Discrete Mathematics: An Open Introduction*, https://discrete.openmathbooks.org/dmoi3/, 3rd edition. License: CC BY-SA: Attribution-ShareAlike.

Sassolas, Mathieu. (2021). Chapters I, II, III, and VI.B. *Introduction to Discrete Mathematics: An OER for MA-471*, https://academicworks.cuny.edu/qb_oers/179/. License: CC BY-NC: Attribution-NonCommercial.