Relations

CISC 2210 — CUNY Brooklyn College
Lecture Notes

Relations

We introduced the idea of **Cartesian products** on slide 18 of Topic 2 and said that they are sets of all the possible ordered pairs of the form (s, t), where $s \in S$ and $t \in T$ (S and T are sets of some kinds of elements.) The notation for a cartesian product is $S \times T$ (`$S \times T$`).

In our chapter, we will present various sets of ordered pairs. In particular, a **relation** (also called a **binary relation**,) denoted by R or some other symbol or name, is a set of ordered pairs of the form (s, t), where $s \in S$ and $t \in T$. It doesn't necessarily contain *all* the possible pairs (s, t). That is, R is a subset of $S \times T$, and not necessarily equal to $S \times T$.

Examples:

- For $S = \{$😀, 😁, 😑, 😒$\}$ and $T = \{$😔, 😄, 😆$\}$, we can create the relation:

  Smiles = {(😀, 😄), (😀, 😆), (😁, 😄), (😁, 😆)}.

- For $\text{Grades} = \{85, 90, 95, 100\}$ and $\text{Letters} = \{B-, B, B+,\ A-, A, A+\}$, we can create the relation:

$$\text{CUNYfirst-Grades} = \{(85, B), (90, A-), (95, A), (100, A+)\}.$$

Relations

More info about relations:

- If some pair (s, t) is in a relation R, we can write: $(s, t) \in R$ (`$(s, t) \in R$`) or, even cooler: sRt (`$s R t$`). If this is the case, we say that s is R-related to t. <u>Examples</u>: (😀, 😆) $\in \mathrm{Smiles}$ and $100\ \mathrm{CUNYfirst\text{-}Grades}\ A+$ (as in sRt).
- We don't need to stop only at relating two sets: we could have multi-set relations. <u>Example</u>: For $S_1 = \{1, 2\}$, $S_2 = \{3, 5\}$, and $S_3 = \{0, 6\}$, we could have the **tertiary** relation: $R = \{(1, 5, 0), (2, 5, 6), (1, 3, 0)\}$ (this is a set of triples!) However, in the scope of this course, we'll focus only on binary relations.
- In a relation from set S to set T, there's no limit on how many times some $s \in S$ is the 1st coordinate in ordered pairs. <u>Example</u>: If $S = \{1, 2\}$ and $T = \{3, 5\}$, we aren't limited to having only one pair that has 1 as its 1st coordinate: you could have two or more pairs starting with 1, e.g., $\{(1, 3), (1, 5)\}$, or zero pairs starting with 1 (that is, there may be no pair that has 1 in it at all!)
- Likewise, there's no limit on how many times some $t \in T$ is the 2nd coordinate in ordered pairs. <u>Example</u>: If $S = \{1, 2\}$ and $T = \{3, 5\}$, we aren't limited to having only one pair that has 5 as its 2nd coordinate: you could have two or more pairs starting with 5, e.g., $\{(1, 5), (2, 5)\}$, or zero pairs ending with 5 (that is, there may be no pair that has 5 in it.)

Relations

More info about relations:

- The sets S and T that create a relation might differ from one another regarding the kinds of elements they have. Example: CUNYfirst-Grades from before is a relation from numerical grades (= integers from 0 to 100) to letter grades. You may also create relations from some familiar sets, e.g., from $\mathbb{Z}$ to $\mathbb{N}$, such as a relation of an integer to its absolute value: $\text{Abs} = \{(-5,5),(-3,3),(0,0),(2,2),(3,3),\ldots\}$. We can, therefore, write that $-5 \text{ Abs } 5$ and $3 \text{ Abs } 3$, for instance.
- You may even relate a set S to itself. Example: The relation Equals, or '$=$', from $\mathbb{P}$ to $\mathbb{P}$ is one that relates a positive integer to itself: $\text{Equals} = \{(1,1),(2,2),(3,3),(4,4),\ldots\}$. We can, therefore, write that $2 \text{ Equals } 2$, or $2 = 2$, for instance. A relation of a set S to itself is shorthandedly called a **relation on S**.
- An example of a relation on $\mathbb{N}$: The LessThanOrEqualTo relation, or '$\leq$', is as follows: $\text{LessThanOrEqualTo} = \{(0,0),(0,1),(0,2),\ldots,(1,1),(1,2),(1,3),\ldots,(2,2),(2,3),\ldots\}$. For example, we see that $1 \leq 1$ and that $2 \leq 3$.
  Bonus question: What's the idea of this relation? E.g., given the pattern above, to which integers will $\mathbf{5}$ relate?

Fun Class Activities:

1. List all the elements (pairs) in the relation Equals that is defined from $S = \{1, 2, 4, 15\}$ to $T = \{2, 5, 15, 21, 32\}$.
2. List all the elements (pairs) in the relation $\mathrm{EvenToOdd}$, which relates even integers to odd integers, that is defined from $S = \{1, 2, 4, 15\}$ to $T = \{2, 5, 15, 21, 32\}$.
3. List all the elements (pairs) in the relation $\mathrm{GreaterThan}$, or $>$, that is defined from $S = \{1, 2, 3, 4\}$ to $S = \{1, 2, 3, 4\}$.
4. Consider the set $S = \{$😃, 😁, 😎, 😜$\}$ and the following relation on S: $R = \{($😃, 😃$)$, $($😁, 😁$)$, $($😎, 😎$)$, $($😜, 😜$)\}$.
   What relation is this?

You can create your own relations between any sets you are working with, and also create or choose a symbol for that relation, e.g., $\approx$ (`$\approx$`), $\bowtie$ (`$\bowtie$`), or even $\heartsuit$ (`$\heartsuit$`). Just make sure to clearly describe what the related sets are, and what exactly the relation is doing; that is, what is the rule by which elements are related?

Any questions so far?

Relation Properties

Some of the same-set relations we interact with have special **properties**:

- **Definition.** [Reflexive relations] A relation R on a set S is called **reflexive** if, for every $x \in S$, the pair (x, x) is in R.
- **Definition.** [Anti-reflexive relations] A relation R on a set S is called **anti-reflexive** if, for every $x \in S$, the pair (x, x) is NOT in R.
- **Definition.** [Symmetric relations] A relation R on a set S is called **symmetric** if, for every pair (x, y) in R (where $x, y \in S$,) the pair (y, x) is also in R.
- **Definition.** [Anti-symmetric relations] A relation R on a set S is called **anti-symmetric** if the following is true: if the pairs (x, y) and (y, x) are both in R, then $x = y$.
- **Definition.** [Transitive relations] A relation R on a set S is called **transitive** if the following is true: if the pairs (x, y) and (y, z) are both in R, then the pair (x, z) is also in R.

Clearly, some of these properties can't exist together for the same relation; for instance, no relation can be both reflexive and anti-reflexive.

Relation Properties

Example: The LessThanOrEqualTo ($\leq$) relation on some set S of comparable objects:

- It is **reflexive** because, for every $x \in S$, we indeed have $x \leq x$, which means that (x, x) is in LessThanOrEqualTo.
- Since it is reflexive, it is NOT anti-reflexive.
- It is NOT symmetric because $x \leq y$ doesn't imply that $y \leq x$. Example: since $2 \leq 3$, we have $(2, 3) \in \text{LessThanOrEqualTo}$. However, it is clear that $3 \not\leq 2$, so $(3, 2) \notin \text{LessThanOrEqualTo}$.
- It is **anti-symmetric** because the only pairs (x, y) and (y, x) that exist in LessThanOrEqualTo are those in which $x = y$. In other words, the relation LessThanOrEqualTo contains pairs of the form (x, x) for which we have $x \leq x$.
- It is **transitive** since if $x \leq y$ and $y \leq z$, it must be that $x \leq z$ (because $x \leq y \leq z$).

In summary, the LessThanOrEqualTo relation has the following 3 properties: reflexive, anti-symmetric, and transitive.

Definition. [Partial Order] A relation that is reflexive, anti-symmetric, and transitive is called a **partial order**.

Relation Properties

Example: The Equals $(=)$ relation on some set S of comparable objects:

- It is **reflexive** because, for every $x \in S$, we indeed have $x = x$, which means that (x, x) is in Equals.
- Since it is reflexive, it is NOT anti-reflexive.
- It is **symmetric** because $x = y$ indeed implies that $y = x$.
- It is **anti-symmetric** because the existence of the pairs (x, y) and (y, x) in Equals means that $x = y$ (and, of course, that $y = x$.)
- It is **transitive** since if $x = y$ and $y = z$, it must be that $x = z$ (because $x = y = z$).

In summary, the Equals relation has the following 4 properties: reflexive, symmetric, anti-symmetric, and transitive.

Definition. [Equivalence Relation] A relation that is reflexive, symmetric, and transitive is called an **equivalence relation**.

The Equals relation is the only anti-symmetric equivalence relation.

Relation Properties

Example: A relation $\mathrm{NotOddToEven}$ on some set S of integers. In this relation, $(x, y) \in \mathrm{NotOddToEven}$ if x and y are both even, or if x and y are both odd, or if x is even and y is odd (but not if x is odd and y is even.)

- It is **reflexive** because, for every $x \in S$, we indeed have $(x, x) \in \mathrm{NotOddToEven}$ since x can be either even (so we have an even-to-even pair) or odd (so we have an odd-to-odd pair.)
- Since it is reflexive, it is NOT anti-reflexive.
- It is NOT symmetric because we have $(2, 3) \in \mathrm{NotOddToEven}$, but $(3, 2) \notin \mathrm{NotOddToEven}$ (since the odd-to-even pair $(3, 2)$ is not allowed.)
- It is NOT anti-symmetric because $(2, 4)$ and $(4, 2)$ are both in $\mathrm{NotOddToEven}$, but $2 \neq 4$.
- It is **transitive** since (x, y) and (y, z) both being inside $\mathrm{NotOddToEven}$ implies that (x, z) is also in $\mathrm{NotOddToEven}$. ***Fun In-Class Activity***: Outline all the possible cases for x, y, and z in terms of eveness and oddness to see that this relation is indeed transitive.

In summary, the $\mathrm{NotOddToEven}$ relation has the following 2 properties: reflexive and transitive.

Relation Properties

Definition. [Congruence] An integer $m \in \mathbb{Z}$ is **congruent** to n modulo p if $m - n$ is a multiple of $p \in \mathbb{Z}$. If so, we write $m \equiv n (\mod p)$ (`$m \equiv n (\mod p)$`). Example: The congruence relation $\mathrm{Congruent}$ on some the set of integers $\mathbb{Z}$.

- It is **reflexive** because, for every $x \in \mathbb{Z}$, we indeed have $(x, x) \in \mathrm{Congruent}$ since $x \equiv x (\mod p)$ because $x - x = 0$ is divisible by any integer p: $\frac{0}{p} = 0$.
- Since it is reflexive, it is NOT anti-reflexive.
- It is **symmetric** because $x \equiv y (\mod p)$ implies that $y \equiv x (\mod p)$ since, if $x - y$ is a multiple of p, then $y - x = -(x - y)$ is a multiple of p, too.
- It is NOT anti-symmetric because, if $x \equiv y (\mod p)$ and $y \equiv x (\mod p)$, it doesn't necessarily mean that $x = y$. For example, if $p = 2$, then $4 \equiv 8 (\mod 2)$ and $8 \equiv 4 (\mod 2)$, but $4 \neq 8$.
- It is **transitive** since $x \equiv y (\mod p)$ and $y \equiv z (\mod p)$ implies that $x \equiv z (\mod p)$. This is because $x - z = (x - y) + (y - z)$ is clearly a multiple of p.

In summary, the $\mathrm{Congruent}$ relation has the following 3 properties: reflexive, symmetric, and transitive.

Relation Properties

Fun Class Activities:

1. What properties does the relation $\mathrm{GreaterThanOrEqualTo}$ ($\geq$) have?
2. What properties does the relation $\mathrm{LessThan}$ ($<$) have?
3. What properties does the relation $\mathrm{EvenToOdd}$ have?
4. Consider the set $S = \{$😃, 😁, 😎, 😜$\}$ and the following relation on S:
   $R = \{($😃, 😃$), ($😁, 😁$), ($😎, 😎$), ($😜, 😜$)\}$.
   What properties does this relation have?
5. Consider the set $S = \{$😃, 😁, 😎, 😜$\}$ and the following relation on S:
   $R = \{($😃, 😃$), ($😁, 😁$), ($😜, 😜$)\}$.
   What properties does this relation have?

Any questions so far?

Converse Relations

Let get back to general relations from some set S to some set T and define a sort of a 'backwards' relation.

Definition. [Converse Relations] Let R be a relation from set S to set T. The **converse relation** of R, denoted by $R^{\leftarrow}$ (`$R^{\leftarrow}$`), contains pairs of the form (t, s) only if R contains the pairs (s, t) (where $t \in T$ and $s \in S$).

Example #1: The converse of the relation $\mathrm{LessThanOrEqualTo}$ ($\leq$) is $\mathrm{GreaterThanOrEqualTo}$ ($\geq$) because, for every pair $x \leq y$, which exists in $\mathrm{LessThanOrEqualTo}$, the pair $y \geq x$ exists in $\mathrm{GreaterThanOrEqualTo}$.

Example #2: The converse of the relation $\mathrm{EvenToOdd}$ is $\mathrm{OddToEven}$ because, for every pair (x, y) with x being even and y being odd, which exists in $\mathrm{EvenToOdd}$, the pair (y, x) exists in $\mathrm{OddToEven}$.

Example #3: The converse of the relation $\mathrm{Smiles} = \{($😀, 😄$), ($😀, 😆$), ($😁, 😄$), ($😁, 😆$)\}$ is:
$\{($😄, 😀$), ($😆, 😀$), ($😄, 😁$), ($😆, 😁$)\}$. (Simply exchange the 1st coordinate with the 2nd coordinate.)

Converse Relations

Fun fact: the converse of a relation R on S (from S to S,) will preserve any of the properties that R has, such as reflexivity, anti-reflexivity, symmetry, anti-symmetry, and transitivity, whichever of them that R has.

Fun Class Activities:

1. What is the converse relation of the relation $\mathrm{LessThan}$ $(<)$?
2. What is the converse relation of the relation Equals $(=)$?
3. What is the converse relation of the relation $\mathrm{CUNYfirst\text{-}Grades}$ from slide 2? (List its elements.)

Converse Relations

Any questions so far?

Equivalence Relations

Definition. [Equivalence Relations] An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

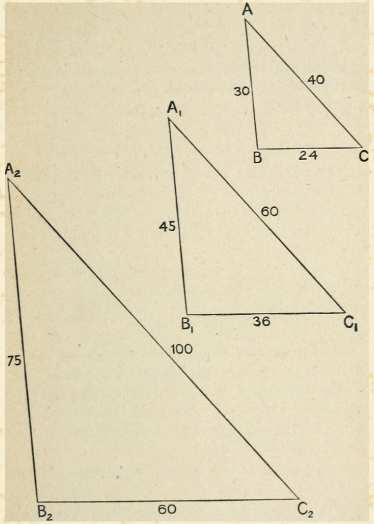
Two objects x and y that are related through an equivalence relation are called **equivalent**. Other synonyms possible are **equal**, **similar**, **congruent**, and **isomorphic** (recall that Equal and Congruent are both equivalence relations!)

Possible symbols for representing equivalence relations that you may see in discrete math books are:

- $=$ ('equal'), as in $x = y$ (`$x = y$`)
- $\equiv$ ('congruent' or 'equivalent'), as in $x \equiv y$ (`$x \equiv y$`)
- $\approx$ ('approximate'), as in $x \approx y$ (`$x \approx y$`)
- $\sim$ ('similar'), as in $x \sim y$ (`$x \sim y$`)
- $\simeq$ ('sim-equal'), as in $x \simeq y$ (`$x \simeq y$`)
- $\cong$ ('congruent'), as in $x \cong y$ (`$x \cong y$`)
- $\leftrightarrow$ ('equivalent', especially from a logic perspective), as in $x \leftrightarrow y$ (`$x \leftrightarrow y$`)

Equivalence Relations

Example #1: Triangles whose angles match up (even if their corresponding sides aren't equal) are called **similar triangles**. The relation that matches similar triangles to one another (let's call it $\mathrm{Sim\text{-}Triangle}$) is an equivalence relation.



Three (3) similar triangles. flickr, CC0 1.0: Public Domain.

Example #2: Triangles whose angles match up and whose corresponding sides are equal are called **congruent triangles** and create a relation (let's call it $\mathrm{Cong\text{-}Triangle}$) that is an equivalence relation, too.

Equivalence Relations

Example #3: Consider the set of rational numbers: $\mathbb{Q}$. Each number can be written in more than one way, e.g., $\frac{1}{2} = \frac{2}{4} = 0.5 = 50 : 100 = 50\%$, etc.

The relation that relates ways of writing out the same number:

$$\text{Same-}\mathbb{Q}\text{-Num} = \{(\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 0.5), (0.5, \tfrac{1}{2}), (0.5, 0.5), (\tfrac{1}{2}, 50 : 100), \ldots\}$$

is an equivalence relation. This is true because:

- For every $x \in \mathbb{Q}$, we have $x = x$ (e.g., $0.5 = 0.5$), so the relation is reflexive.
- Also, since $x = y$ (e.g., $\frac{1}{2} = 0.5$) implies that $y = x$ ($0.5 = \frac{1}{2}$) for any $x, y \in \mathbb{Q}$, this relation is also symmetric.
- Finally, for any $x, y, z \in \mathbb{Q}$, if $x = y$ (e.g., $\frac{1}{2} = 0.5$,) and $y = z$ (e.g., $0.5 = 50\%$,) this means that $x = z$ (so $\frac{1}{2} = 50\%$,) so the relation is transitive.

Equivalence Relations

Example #4: Consider the set of rational numbers $\mathbb{Q}$ that are written as decimals, e.g., 0.5, 1, 1.25, -15.67, etc.

Definition. [Integer Part] The **integer part** of a decimal number of the form $m.n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, is the integer m. Notation: the integer part of a positive $k \in \mathbb{Q}$ is denoted by $\lfloor k \rfloor$ (`$\lfloor k \rfloor$`) and called the **floor** of k.

Examples: $\lfloor 0.5 \rfloor = 0$, $\lfloor 1 \rfloor = 1$, and $\lfloor 1.25 \rfloor = 1$. However, $\lfloor -15.67 \rfloor = -16$.

The relation that relates every two rational numbers whose integer parts are the same is an equivalence relation:

$$\text{Same-Int-Parts} = \{(0, 0.5), (0, 0), (0.25, 0.5), \ldots, (1, 1), (1, 1.25), \ldots\}.$$

Definition. [Ceiling] The operation of rounding up a real number $x \in \mathbb{R}$ is called **taking the ceiling** of x, or **ceil** shortly. Notation: the ceil of x is denoted by $\lceil x \rceil$ (`$\lceil x \rceil$`).

Examples: $\lceil 0.5 \rceil = 1$, $\lceil 1 \rceil = 1$, $\lceil 1.25 \rceil = 2$, and $\lceil -15.67 \rceil = -15$ (so it is ceil that finds the integer part of negative nums!)

Equivalence Relations

The following is an example of a relation that isn't an equivalence relation:

Consider the set of all the entries in a certain dictionary. Some entries represent words that are synonyms of one another.

Examples: The word bank has many synonyms, some of which are $\mathrm{fund},\ \mathrm{stock},\ \mathrm{savings},\ \mathrm{side},\ \mathrm{slope},\ \mathrm{coast},\ \mathrm{riverside}.$

Consider the relation that relates every two synonym words:

$$\mathrm{Synonyms} = \{(\mathrm{bank}, \mathrm{bank}), (\mathrm{bank}, \mathrm{savings}), (\mathrm{savings}, \mathrm{bank}), (\mathrm{savings}, \mathrm{savings})\ (\mathrm{bank}, \mathrm{riverside}), \\ (\mathrm{riverside}, \mathrm{bank}), (\mathrm{riverside}, \mathrm{riverside}), \ldots\}.$$

This relation is reflexive and symmetric, but it isn't transitive. Example: We have $(\mathrm{savings}, \mathrm{bank}) \in \mathrm{Synonyms}$ and $(\mathrm{bank}, \mathrm{riverside}) \in \mathrm{Synonyms}$ but $(\mathrm{savings}, \mathrm{riverside}) \notin \mathrm{Synonyms}$ because savings isn't a synonym of $\mathrm{riverside}$. Since $\mathrm{Synonyms}$ isn't transitive, it isn't an equivalence relation.

Equivalence Relations

Fun Class Activities:

1. **Definition**. [Fraction Part] The **fraction part** of a decimal number of the form $m.n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, is the rational number $0.n$.

   Example #1: the fraction part of -15.67 is 0.67.
   Example #2: the fraction part of $1 = 1.0$ is $0.0 = 0$.

   Let's call the relation that relates every two rational numbers whose fraction part are the same $\mathrm{Same\text{-}Frac\text{-}Parts}$. Is this relation an equivalence relation? Explain.
2. Is the $\mathrm{Sibling}$ relation (that we had in one of our polls) an equivalence relation? Explain.
3. Is the $\mathrm{LessThanOrEqualTo}$ $(\leq)$ relation an equivalence relation? Explain.

Any questions so far?

Equivalence Classes

Definition. [Equivalence Classes] Let R be an equivalence relation on a set S, and consider some object $x \in S$. The set

$$[x] = \{y \in S \mid (x, y) \in R\}$$

(`$[x] = \{y \in S \mid (x, y) \in R\}$`) (that is, the set of all the elements y of S that relate to x through R) is called the **equivalence class**, or **equivalence set**, containing x.

Example: In the relation $\mathrm{Same}\text{-}\mathbb{Q}\text{-}\mathrm{Num}$ from slide 19, since we have $\frac{1}{2} = \frac{2}{4} = 0.5 = 50 : 100 = 50\%$, we could write:

$$[\tfrac{1}{2}] = \{\tfrac{1}{2}, \tfrac{2}{4}, 0.5, 50 : 100, 50\%, \ldots\}$$

You can also write $[0.5]$, $[50\%]$, $[50 : 100]$, etc., instead of $[\frac{1}{2}]$.

Equivalence Classes

Definition. [A set's partition] Consider a non-empty set S. A **partition** on S is the collection (= set) of non-empty subsets $S_1, S_2, S_3, \ldots, S_n$ of S such that both of the following is true:

1. Every two sets S_i and S_j are **disjoint**: they share no common elements, and
2. The union of all the subsets is S itself: $\bigcup S_i = S$.

Example: Let $S = \{0,\ 0.25,\ 0.5,\ 1,\ 1.5,\ -15,\ -15.67\}$. Each of the following is a partition on S:

- $P_1 = \{\{0, 1, -15\}, \{0.25, 0.5, 1.5, -15.67\}\}$.
- $P_2 = \{\{0\}, \{0.25\}, \{0.5\}, \{1\}, \{1.5\}, \{-15\}, \{-15.67\}\}$.
- $P_3 = \{\{0, 0.25, 0.5\}, \{1, 1.5\}, \{-15, -15.67\}\}$.

There are many other ways in which we can partition S.

Equivalence Classes

The following sets, however, aren't partitions of S:

- $NP_1 = \{\{0, 1, -15\}, \{0.25, 0.5, 1, 1.5, -15.67\}\}$ because 1 appears in more than one subset.
- $NP_2 = \{\{0\}, \{0.25\}, \{1\}, \{1.5\}, \{-15\}, \{-15.67\}\}$ because 0.5 is missing.
- $NP_3 = \{\{0, 0.25, 0.5\}, \{\}, \{1, 1.5\}, \{-15, -15.67\}\}$ because the 2nd item is an empty set $\{\}$, but partitions mustn't contain empty sets.

Theorem. [Equivalence classes vs. partitions] Let R be an equivalence relation on a set S. The following are true:

1. The set of all the equivalence classes under R is a partition on S.
2. Every partition on S corresponds to some equivalence relation on S.

Equivalence Classes

Examples:

- $P_1 = \{[0], [0.25]\} = \{\{0, 1, -15\}, \{0.25, 0.5, 1.5, -15.67\}\}$ would correspond to an equivalence relation that relates integers to integers, and fractions to fractions. We can call it Int-vs-Frac.
- $P_2 = \{\{0\}, \{0.25\}, \{0.5\}, \{1\}, \{1.5\}, \{-15\}, \{-15.67\}\}$ corresponds to the Equals relation on S.
- $P_3 = \{[0], [1], [-15]\} = \{\{0, 0.25, 0.5\}, \{1, 1.5\}, \{-15, -15.67\}\}$ corresponds to the Same-Int-Parts relation on S.

Fun Class Activity: Let $S = \{1, 1.23, 2, 2.0, 2.5, 3, \frac{9}{3}\}$.

1. Write out the partition on S that corresponds to the relation $\text{Same-}\mathbb{Q}\text{-Num}$.
2. What relation does the following partition of S correspond to: $\{\{1, 1.23\}, \{2, 2.0, 2.5\}, \{3, \frac{9}{3}\}\}$?

Equivalence Classes

Any questions so far?

Relations: Sources

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