Functions

CISC 2210 — CUNY Brooklyn College
Lecture Notes

Functions

So, what's a function?

Definition. [Function] A **function** is a relation from set D to set R that assigns to each element in D exactly one element in R.

Example: Suppose that $3 \in D$. A function from D to R will contain one pair of the form $(3, y)$, where y is exactly one element in R. For example, the function might have $(3, 2)$. But there won't be additional pairs of the form $(3, y)$ in the function in which $y \neq 2$. For example, a function can't have both $(3, 2)$ and $(3, 0)$ in it.

Definition. [Domain] The set D of **inputs** into a function is called the **domain**: this is the set of first coordinates x of the pairs (x, y) in the function.

Definition. [Range] The set R of **outputs** from a function is called the **range**: this is the set of second coordinates y of the pairs (x, y) in the function.

Definition. [Codomain] A perhaps larger set C that contains the range set R is called the **codomain** of the function.

If a function is named f, and the domain element is x, then the corresponding range element is written as $f(x)$ [read as: f of x – "eff of ex"], which is also called the **image** of x.

Example: Consider the function $f(x) = 2x$ and the domain $D = \{1, 2, 3\}$. The range/image under this function is $R = \{2, 4, 6\}$ (because the function takes each element in D and outputs its product by 2.) That is, we have $f(1) = 2 \cdot 1 = 2$, $f(2) = 2 \cdot 2 = 4$, and $f(3) = 2 \cdot 3 = 6$. Also, since all the elements in the range R are integers, we can say that the codomain is $\mathbb{P}$, $\mathbb{N}$, or $\mathbb{Z}$, depending on how much precise we want to be.

As a shorthand, instead of writing long verbal descriptions of a function's domain and range/codomain, we could use the following notation instead:

$f : D \to R$ (`$f: D \to R$`) or $f : D \to C$ (`$f: D \to C$`) [We read it as: "f is a function from set D to set R."]

Functions

Example #1: You may be given a function like $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$. Here, both the domain and the codomain are the set $\mathbb{R}$ (**Remember what $\mathbb{R}$ stands for?**)

Example #2: You may be given a function like $f : \mathbb{Z} \to \{-1, 0, 1\}$ such that $f(x) = \mathrm{Sign}(x)$. Here, the domain is the set of all integers, and the range is the set $\{-1, 0, 1\}$.
This function takes an integer and outputs a number as follows: if the input is 0, the output is 0; if the input is positive, the output is 1; and if the input is negative, the output is -1. For simplicity, we can say that $f : \mathbb{Z} \to \mathbb{Z}$ (= the codomain is $\mathbb{Z}$, which definitely contains $\{-1, 0, 1\}$.)

Example #3: You may be given a function like $f : \mathbb{N} \to \{n \mid n = 2k, k \in \mathbb{N}\}$ such that $f(x) = 2x$. Here, the domain is the set of natural integers, and the range is the set of even, non-negative integers (because when we do $2x$ where x is an integer, we get an even integer.)
For simplicity, we can say that $f : \mathbb{N} \to \mathbb{N}$ (= the codomain is $\mathbb{N}$.)

Functions

Fun Class Activities:

1. Is the following relation considered a function: $R_1 = \{(0,1),(1,-2),(0,2),(-1,1)\}$? Why or why not?
2. Is the following relation considered a function: $R_2 = \{(2,1),(3,1),(1,0),(4,1)\}$? Why or why not?
3. Let $f(x) = x + 3$. Find the following values of f:
   a. $f(2)$
   b. $f(0.5)$
   c. $f(-1)$
4. Consider the function $g(x) = x^3$ with the domain $\mathbb{P}$.
   a. What is the **range** of g? List the first 4 elements of the range.
   b. What is a convenient **codomain** for g?
   c. Describe g's domain and codomain using the "$f : D \to C$" notation.

Functions

Any questions so far?

Ways of Describing Functions

We can describe a function in one of several ways. We've already seen a few such ways:

1. By describing the action of the function verbally. For example: "a function f takes positive integers and squares them."
2. By using a math expression, also called **formula**, e.g., $f(x) = x^2$.
   This method is unambiguous, very precise, and powerful: to find the output for a certain input x, we simply plug x into the formula of f and get the output immediately.
3. By listing the pairs (x, y) of the function, e.g., $f = \{(0,0), (1,1), (2,4), (3,9), \ldots\}$.

Other ways of describing functions are:

4. By listing the coordinates of the points of the function in a table, e.g.:

x	0	1	2	3	4	5	6	7	$\ldots$
$f(x)$	0	1	4	9	16	25	36	49	$\ldots$

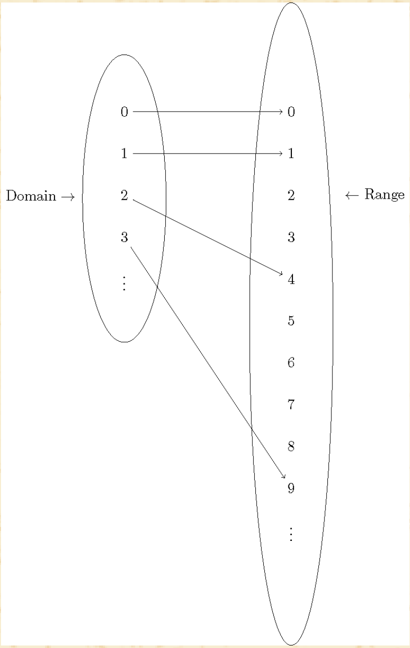
Ways of Describing Functions

5. By creating a **mapping diagram** of the function. The mapping diagram is a sketch of two sets, the domain and the range, in the form of "bubbles".

   Each of the bubbles contains the respective elements of the domain and the range, and arrows are drawn from the elements of the domain into the elements of the range. These arrows represent the points/pairs contained in the function.

   Like the table method, the mapping diagram method can't be used to list a large number of pairs due to space limitations.

   The $\LaTeX$ code for the creation of the diagram on this slide is listed on the next slide.



The mapping diagram for $f(x) = x^2$.
Miriam Briskman, CC BY-NC 4.0

Ways of Describing Functions


```
\documentclass{standalone}
\usepackage{tikz} % A graphics library
\usetikzlibrary{arrows, shapes, fit}

\begin{document}

\begin{tikzpicture}
  % Define elements in the domain
  \node (d0) at (0,0) {0};
  \node (d1) at (0,-1) {1};
  \node (d2) at (0,-2) {2};
  \node (d3) at (0,-3) {3};
  \node (dcont) at (0,-4) {$\vdots$};
```

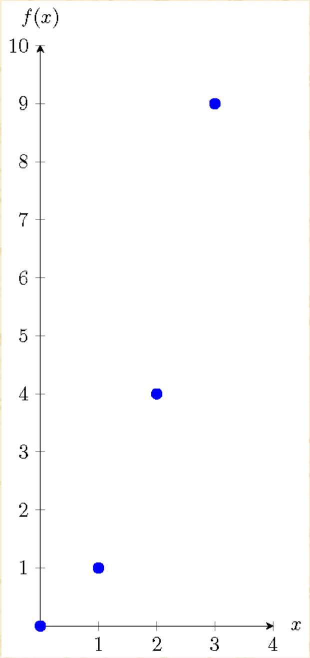
Ways of Describing Functions

6. By drawing the **graph** of the function on the **coordinate plane** (= a 2D surface whose horizontal dimension stands for x and the vertical dimension stands for $f(x) = y$.)

   Each of the points in the function is **plotted** (marked) on the graph as a dot (if the function is discrete) or as a solid line or curve (if the function is continuous.)

   You can plot multiple functions on the same plane (and use this to compare the graphs of the functions, e.g., which function grows faster.)

   The $\LaTeX$ code for the creation of the graph on this slide is listed on the next slide.



The graph for $f(x) = x^2$. Miriam Briskman, CC BY-NC 4.0

Ways of Describing Functions

```
\documentclass{standalone}
\usepackage{pgfplots} % Plotting library

\begin{document}

\begin{tikzpicture}
    \begin{axis}[
        xlabel = $x$,
        ylabel = $f(x)$,
        y = 1cm,
        x = 1cm,
        xmin = 0, xmax = 4, ymin = 0, ymax = 10,
        axis lines = middle,
        xlabel style = {anchor = west, xshift = 5pt},
```

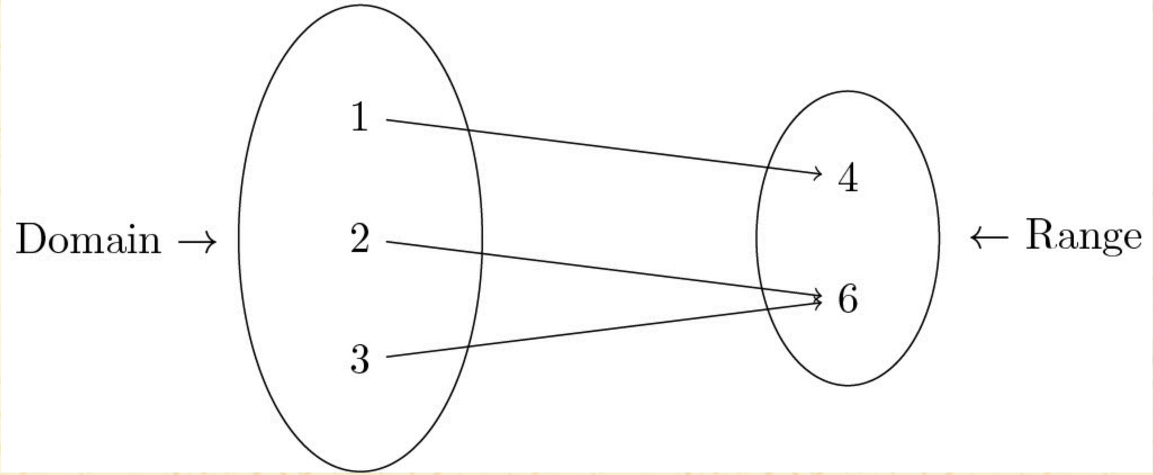
Ways of Describing Functions

Any questions so far?

Function Or Not: Unique Arrows Test

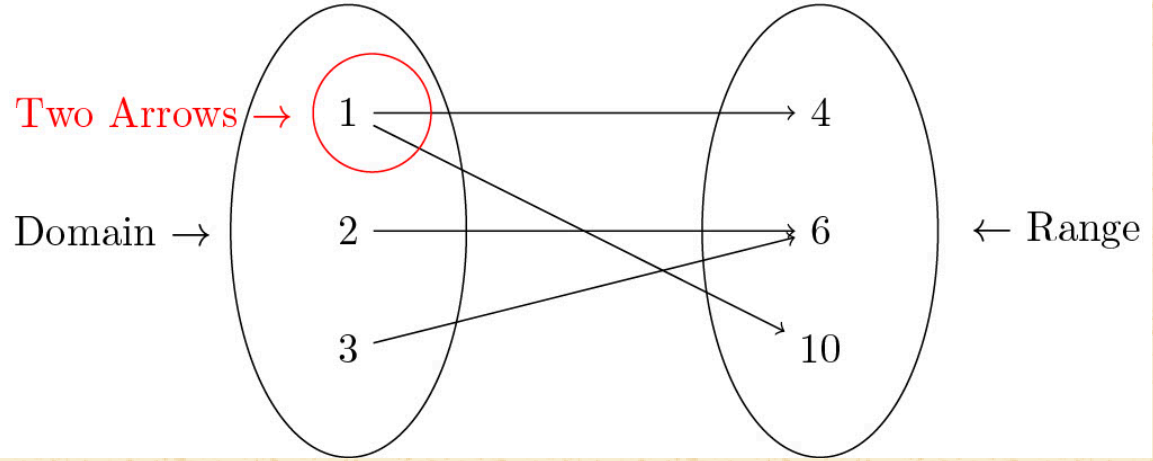
A mapping diagram can easily tell us whether the shown relation is a function or not. We call the rule that tells this the **Unique Arrows Test**: if only one arrow comes from every domain element, then the relation is a function; otherwise, if two or more arrows come from a single domain element, then the relation is NOT a function.

Only one arrow exists from every domain item - a function ✓



This is a function! Miriam Briskman, CC BY-NC 4.0

Two arrows exist from the same domain item - NOT a function ✗



This is NOT a function. Miriam Briskman, CC BY-NC 4.0

Function Or Not: Unique Arrows Test

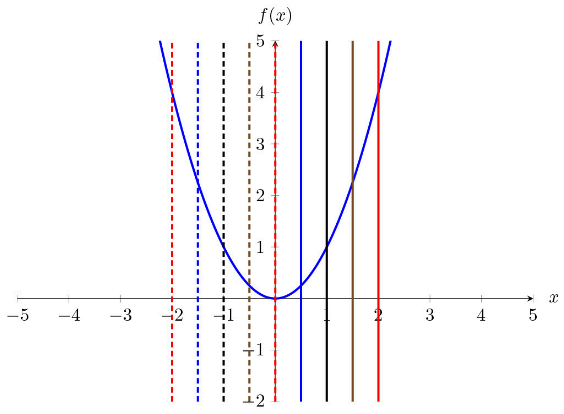

```
% Code for the left diagram:
\documentclass{standalone}
\usepackage{tikz}
\usetikzlibrary{arrows, shapes, fit}

\begin{document}

\begin{tikzpicture}
  % Define elements in the domain
  \node (d0) at (0,0) {1};
  \node (d1) at (0,-1) {2};
  \node (d2) at (0,-2) {3};

  % Define elements in the range
```

```
% Code for the right diagram:
\documentclass{standalone}
\usepackage{tikz}
\usetikzlibrary{arrows, shapes, fit}

\begin{document}

\begin{tikzpicture}
  % Define elements in the domain
  \node (d0) at (0,0) {1};
  \node (d1) at (0,-1) {2};
  \node (d2) at (0,-2) {3};

  % Define elements in the range
```

Function Or Not: Vertical Line Test

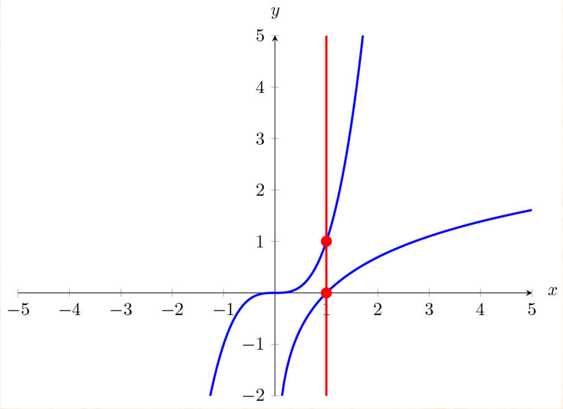
A graph, too, can easily tell us whether the shown relation is a function or not. This rule is called the **Vertical Line Test**: if any vertical line that you can draw on top of the graph intersects only once with the curve, then the relation is a function; otherwise, if you can draw a vertical line that intersects twice or more with the curve, then the relation is NOT a function.

Only one intersection with any vertical line - a function ✓



This is a function! Miriam Briskman, CC BY-NC 4.0

Two or more intersections with a vertical line - NOT a function ✗



This is NOT a function. Miriam Briskman, CC BY-NC 4.0

Function Or Not: Vertical Line Test


```
% Code for the left graph:
\documentclass{standalone}
\usepackage{pgfplots} % Plotting library

\begin{document}

\begin{tikzpicture}
    \begin{axis}[
        xlabel = $x$,
        ylabel = $f(x)$,
        y = 1cm,
        x = 1cm,
        xmin = -5, xmax = 5, ymin = -2, ymax =
5,
```

```
% Code for the right graph:
\documentclass{standalone}
\usepackage{pgfplots} % Plotting library
\usetikzlibrary{intersections}

\begin{document}

\begin{tikzpicture}
    \begin{axis}[
        xlabel = $x$,
        ylabel = $y$,
        y = 1cm,
        x = 1cm,
        xmin = -5, xmax = 5, ymin = -2, ymax =
```

Function Or Not: Vertical Line Test

Any questions so far?

Function Properties: One-To-One

Definition. [One-To-One Functions] A function is called **one-to-one** or **injective** if different elements in the domain always map to different elements in the range. Symbolically, a function f is injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Equivalently, in a one-to-one function, if $x_1 \neq x_2$, then $f(x_1)$ is never equal to $f(x_2)$.

For example, the function $f(x) = 2x + 3$ from $\mathbb{R} \to \mathbb{R}$ is injective.

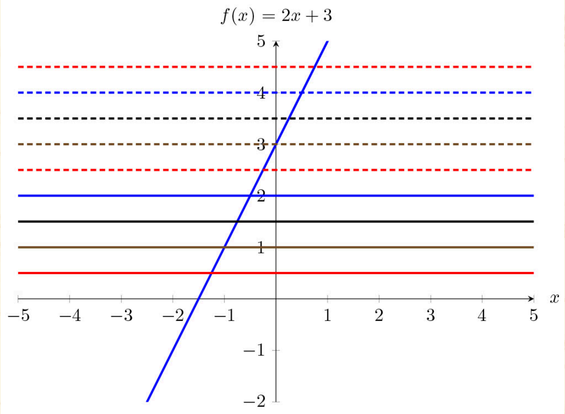
Not every function is injective, however. The function $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is not injective, because $f(2) = f(-2)$ (and, clearly, $2 \neq -2$.)

Injectivity ensures that no two domain elements share the same output (= range) value.

One-To-One Or Not: Horizontal Line Test

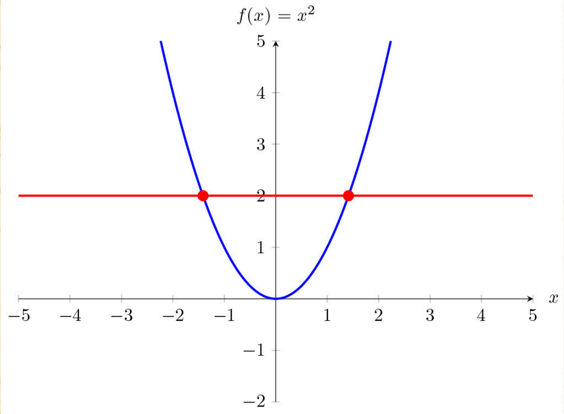
A graph can also tell us if a given function has a one-to-one property or not. This rule is called the **Horizontal Line Test**: if any horizontal line that you draw on top of the graph intersects at most once with the curve, then the function is injective; otherwise, if you can draw a horizontal line that intersects twice or more with the curve, then the function is NOT one-to-one.

Only one intersection with any horizontal line: injective ✓



This is a one-to-one function! Miriam Briskman, CC BY-NC 4.0

Two or more intersections with a horizontal line: NOT injective ✗



This is NOT a one-to-one function. Miriam Briskman, CC BY-NC 4.0

One-To-One Or Not: Horizontal Line Test


```
% Code for the left graph:
\documentclass{standalone}
\usepackage{pgfplots} % Plotting library

\begin{document}

\begin{tikzpicture}
    \begin{axis}[
        xlabel = $x$,
        ylabel = $f(x)\ {=}\ 2x+3$,
        y = 1cm,
        x = 1cm,
        xmin = -5, xmax = 5, ymin = -2, ymax =
5,
```

```
% Code for the right graph:
\documentclass{standalone}
\usepackage{pgfplots} % Plotting library
\usetikzlibrary{intersections}

\begin{document}

\begin{tikzpicture}
    \begin{axis}[
        xlabel = $x$,
        ylabel = $f(x)\ {=}\ x^2$,
        y = 1cm,
        x = 1cm,
        xmin = -5, xmax = 5, ymin = -2, ymax =
```

Function Properties: One-To-One

We can numerically check if a function is one-to-one or not as follows:

1. Set up an equation of the form $f(x_1) = f(x_2)$. For instance, for $f(x) = 2x + 3$, we will do: $2x_1 + 3 = 2x_2 + 3$.
2. Solve the equation for x_1. We do the following:
$$2x_1 + 3 = 2x_2 + 3 \text{ [Original equation]}$$
$$2x_1 = 2x_2 \text{ [Subtracting 3 from each side]}$$
$$x_1 = x_2 \text{ [Dividing each side by 2]}.$$
3. If you got $x_1 = x_2$ as the only solution to the equation, then the function f is one-to-one. In our example for $f(x) = 2x + 3$, we indeed found that f is injective.

Contrastingly, the function $f(x) = x^2$ is not injective because we get $(x_1)^2 = (x_2)^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$. In particular, because we got a solution that isn't $x_1 = x_2$ (we also got $x_1 = -x_2$,) this means that $f(x) = x^2$ is not injective.

Function Properties: One-To-One

Any questions so far?

Function Properties: Onto

Definition. [Onto Functions] A function is called **onto** or **surjective** if every element in the codomain is the output of at least one element in the domain. That is, for a function $f : A \to B$, f is surjective if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

To prove surjectivity, one must show how to solve the equation $f(x) = y$ for x given an arbitrary y in the codomain. For example, the function $f(x) = 2x + 3$ from $\mathbb{R} \to \mathbb{R}$ is surjective, because solving for x gives us: $x = \frac{y-3}{2}$. This number ($\frac{y-3}{2}$) is a real number (that is, x is in the domain of f for any $y \in R$.)

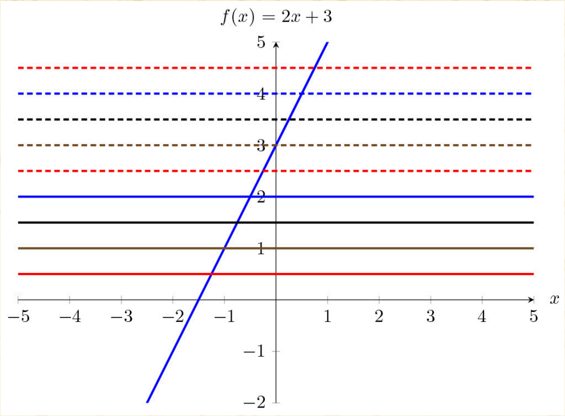
However, the function $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is not surjective, because no negative number is a square. Specifically, if we pick $y = -1$, we get: $x^2 = -1 \Rightarrow x = \sqrt{-1}$, which isn't a real number (it's the complex number $i = \sqrt{-1}$.)
Since the x we got isn't in f's domain ($\sqrt{-1} \notin \mathbb{R}$), we understand that $f(x) = x^2$ is not 'onto'.

Surjectivity guarantees that the range of the function equals the entire codomain.

Onto Or Not: Horizontal Line Test

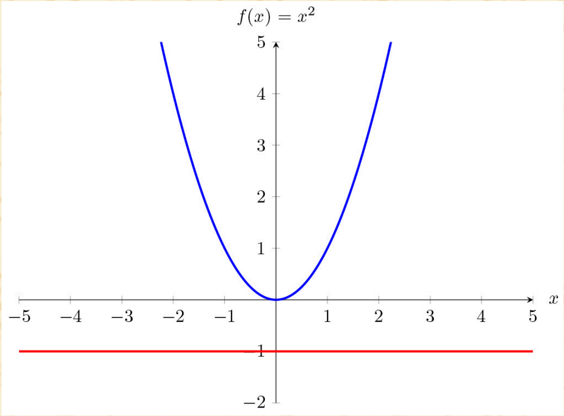
The Horizontal Line Test can also be used to tell if a function is or isn't onto: if any horizontal line that you draw on top of the graph intersects at least once with the curve, then the function is surjective; otherwise, if you can draw a horizontal line that doesn't intersect at all with the curve, then the function is NOT onto.

Only one intersection with any horizontal line: surjective ✓



This is an onto function! Miriam Briskman, CC BY-NC 4.0

No intersections with a horizontal line at all: NOT surjective ✗



This is NOT an onto function. Miriam Briskman, CC BY-NC 4.0

Function Properties: Onto

Any questions so far?

Function Properties: One-To-One Correspondence

Definition. [One-To-One Correspondence Functions] A function is called **bijective** (also known as **one-to-one correspondence**) if it is both injective and surjective.

This means every element in the codomain is mapped to by exactly one element from the domain, and vice versa.

For example, the function $f(x) = x + 5$ from $\mathbb{R} \to \mathbb{R}$ is bijective, since it is both injective and surjective.

From the previous slides, we also now know that $f(x) = 2x + 3$ is a **bijection**, while $f(x) = x^2$ isn't.

In bijective functions, there is a perfect pairing between domain and codomain elements.

Only bijective functions have well-defined **inverses**: we'll cover inverses on the next slide.

Function Properties: Inverses

Definition. [Function's Inverse] The **inverse** of a function f, denoted f^{-1}, reverses the mapping of f.

This is the exact same concept as for converse relations, in which we took every point (x, y) in the relation and flip it into (y, x) to get the converse.

Theorem. [Inverse] The function f has an **inverse** if and only if f is a bijection. That is, if a function is not bijective, its inverse does not exist as a function.

If $f : A \to B$ is a bijection, then the inverse $f^{-1} : B \to A$ satisfies $f^{-1}(f(x)) = x$ for all $x \in A$. Also, $f(f^{-1}(y)) = y$ for all $y \in B$.

For example, if $f(x) = 3x + 2$, then its inverse is $f^{-1}(y) = \frac{y-2}{3}$. Note that $f^{-1}(f(x)) = \frac{(3x+2)-2}{3} = \frac{3x}{3} = x$.

An inverse function "undoes" the action of the original function.

Function Properties: Function Composition

Definition. [Function Composition] The **composition** of two functions f and g is written $f \circ g$ (`$f \circ g$`), which means applying g first and then f.

If $f : B \to C$ and $g : A \to B$, then the composition $f \circ g : A \to C$ is defined by $(f \circ g)(x) = f(g(x))$.

For example, if $f(x) = x^2$ and $g(x) = x + 1$, then $(f \circ g)(x) = f(g(x)) = (x + 1)^2$.

Function composition is not necessarily commutative, meaning $f \circ g \neq g \circ f$ in general. For example, if $f(x) = x^2$ and $g(x) = x + 1$ again, then $(g \circ f)(x) = g(f(x)) = x^2 + 1 \neq (x + 1)^2 = f(g(x)) = (f \circ g)(x)$.

The domain of the composition $f \circ g$ is the set of all x such that $g(x)$ is in the domain of f.

Function composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

Any questions so far?

Functions: Sources

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