

Counting & Combinatorics

CISC 2210 — CUNY Brooklyn College

Lecture Notes



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Counting & Combinatorics: Intro

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Counting and **combinatorics** are important areas of discrete mathematics that study ways to count and arrange objects.

We use counting principles to solve problems involving arrangements, selections, and groupings.

Combinatorics helps computer scientists analyze algorithms and understand data structures. In addition, as we'll see in the next Topic, counting is also the basis for computing useful probabilities of various events.

Counting and combinatorics also provide tools for analyzing advanced discrete structures like graphs.

Key ideas in counting include the **Addition Principle**, **Multiplication Principle**, **permutations**, and **combinations**.

More advanced ideas include the **Binomial Coefficient**, **Inclusion-Exclusion Principle**, and the **Pigeonhole Principle**.



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Addition Principle

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The **Addition Principle** says that if one task can be done in m ways and another task in n ways, then there are $m + n$ ways to do either task.

We use the Addition Principle when choices are mutually exclusive and cannot happen at the same time.

Examples:

- If there are 3 different cakes and 4 different mousses on the menu, you have $3 + 4 = 7$ total dessert choices.
- A college student is required to take 1 college class from a group of 5 math and 3 computer science courses, so there are $5 + 3 = 8$ options in total.

It is important that the choices are separate and do not overlap.

If choices can overlap, we will need to adjust our counting to avoid double-counting: see the next slide.



Inclusion-Exclusion (Subtraction) Principle

The **Inclusion-Exclusion Principle** helps count elements correctly when sets overlap.

For two sets the following is always true, $|A \cup B| = |A| + |B| - |A \cap B|$.

Examples:

- If 30 students are taking CISC 2210, 25 are taking CISC 3115, and 10 are taking both, then $30 + 25 - 10 = 45$ students take either of these 2 classes.
- If 15 people like pizza, 12 like burgers, and 5 like both, then $15 + 12 - 5 = 22$ people like pizza or burgers.

We subtract the overlap because it was counted twice.

With three or more sets, more complicated versions of Inclusion-Exclusion are used.



Inclusion-Exclusion Principle on 3 Sets

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Scenario: Suppose that, in a group of students, 30 study Math, 25 study Physics, and 20 study CS. Also, we are told that 12 students study both Math and Physics, 10 study both Math and CS, and 8 study both Physics and CS. Finally, 5 students study all three subjects.

We want to find how many students study at least one subject. Using the Inclusion-Exclusion Principle, we calculate:

$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|.$$

Once we substitute the values, we get:

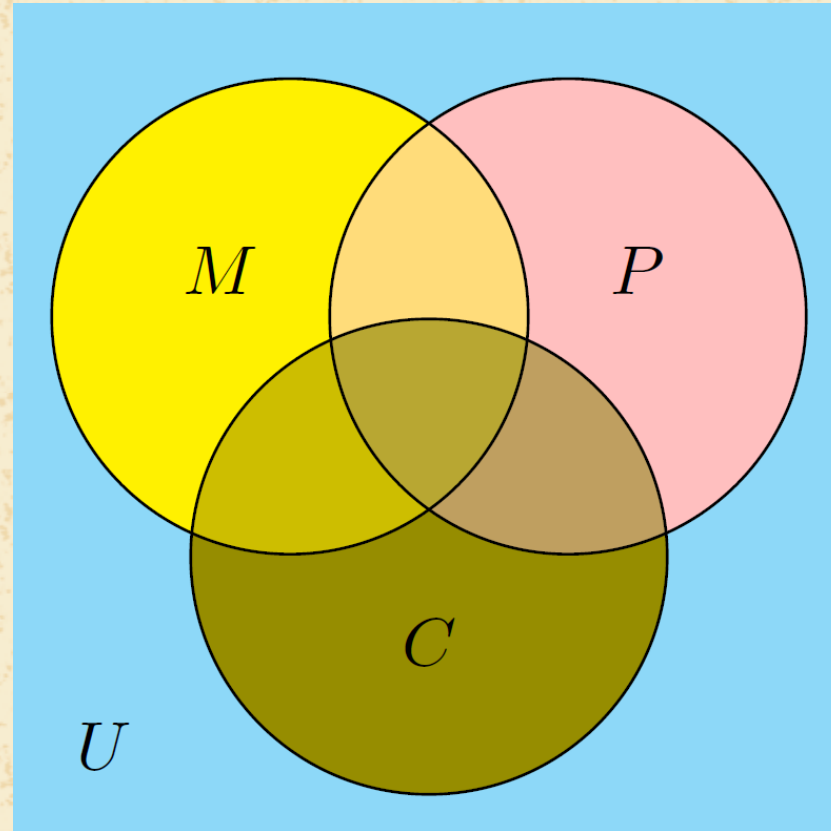
$$|M \cup P \cup C| = 30 + 25 + 20 - 12 - 10 - 8 + 5 = 50.$$

Thus, there are 50 students who study at least one of the three subjects.



Inclusion-Exclusion Principle on 3 Sets

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A venn diagram on three sets, M , P , and C . [Miriam Briskman](#), [CC BY-NC 4.0](#).

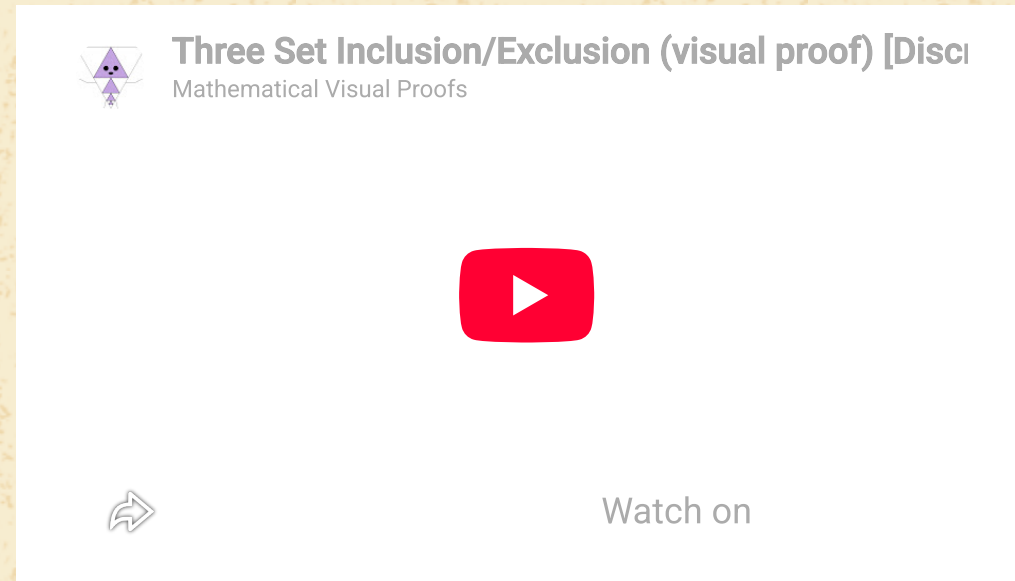


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Inclusion-Exclusion Principle on 3 Sets

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The video at <https://www.youtube.com/watch?v=vVZwe3TCJT8> illustrates why the IE Principle on 3 Sets actually works:



<https://www.youtube.com/watch?v=vVZwe3TCJT8>



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Multiplication Principle

The **Multiplication Principle** says that if one task can be done in m ways and a second task in n ways, then both tasks together can be done in $m \times n$ ways.

We use the Multiplication Principle when tasks are performed in sequence.

Examples:

- If you have 4 shirts and 3 pairs of pants, you have $4 \times 3 = 12$ different outfits.
- A password has 2 letters (26 options each) followed by 2 digits (10 options each), so there are $26^2 \times 10^2$ total passwords.

The principle assumes that each task is independent of the others.

We often apply the Multiplication Principle to problems involving ordered selections.



Fun Class Activities:

1. Choose between 3 brands of laptops, 6 brands of iPads, or 4 brands of tablets. How many choices do you have?
2. You can paint your room either 6 colors or wallpaper it with 9 styles. How many ways to decorate do you have?
3. You roll a 6-sided die and then flip a 2-sided coin. How many different outcomes can you get?
4. Choose 1 of 4 smartphone colors and 1 of 7 phone case styles. How many combinations do you have?
5. In your neighborhood, 50 people own dogs, 40 own cats, and 20 own both. How many own a dog or a cat?
6. In your school, 15 students play soccer, 10 play basketball, and 5 play both. How many play either sport?
7. In a tech company, 30 employees know Java, 25 know Python, and 10 know both. How many know Java or Python?
8. You posted a short poll in your class about your classmates' favorite ice-cream flavor: chocolate, vanilla, or strawberry. The results were as follows: 31 students preferred chocolate, 18 preferred vanilla, 15 preferred strawberry. Also, 13 preferred both chocolate and vanilla, 12 preferred both chocolate and strawberry, 9 preferred both vanilla and strawberry, and 6 preferred all 3 flavors. How many classmates do you have in that class?



Any questions so far?



A **permutation** of n distinct objects is an ordered arrangement of these objects.

The number of permutations is the **factorial**: $n! = n \times (n - 1) \times \cdots \times 3 \times 2 \times 1$
($n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$).

Examples:

- There are $4! = 24$ ways to arrange 4 different books on a shelf.
- There are $5! = 120$ ways to assign 5 different tasks to 5 employees.

In permutations, the order of all the objects matters very much.

If order of all or some of the objects does not matter to us, we must use combinations instead, as we do on the next slide.



A **combination** is a selection of objects where order does not matter.

The number of combinations of n objects taken r at a time is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ ($\text{\texttt{\$}\binom{n}{r} = \frac{n!}{r!(n-r)!}\text{\texttt{\$}}$).

The term $\binom{n}{r}$ is called the **binomial coefficient** and is read as " n choose r ".

Examples:

- There are $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = \frac{120}{12} = 10$ ways to choose 2 students from 5 students.
- There are $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{5040}{6 \cdot 24} = \frac{5040}{144} = 35$ ways to form a committee of 3 people from 7 candidates.

Combinations are used when the order of selection does not affect the result.

Choosing people for a team usually involves combinations, not permutations.



Permutations of Non-Distinct Objects

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When objects are not distinct, the number of permutations is adjusted by dividing by the factorials of repeated items.

The formula is $\frac{n!}{n_1!n_2!\cdots n_k!}$, where n_i are the counts of repeated objects.

Examples:

- The word LEVEL has $\frac{5!}{2!2!} = 30$ different permutations (since the letter L repeats twice and the letter E repeats twice.)
- The word MISSISSIPPI has $\frac{11!}{4!4!2!} = 34650$ permutations (since the letter I repeats 4 times, the letter S repeats 4 times, and the letter P repeats twice.)

This formula corrects for overcounting identical objects (since L^{blue}LEVEL = L^{red}LEVEL).

In other words, whenever repetition occurs, we must adjust the usual $n!$ formula by dividing by the number of identical objects.



Fun Class Activities:

1. Arrange the letters A, B, C in all possible orders (e.g., ABC, ACB, BAC, etc.) How many ways are there?
2. A race with 5 runners: how many ways can they finish first, second, and third?
3. How many ways can 6 people sit in a row?
4. Choose 4 cards from a deck of 52 cards. How many ways?
5. Pick 3 toppings for a pizza from 10 available toppings. How many ways?
6. How many distinct permutations of BANANA are there?
7. How many distinct ways to rearrange SUCCESS are there?
8. Arrange PEEP in distinct orders.



Any questions so far?



The Binomial Formula

The **Binomial Formula**, which uses the binomial coefficient $\binom{n}{k}$, can be used to multiply the expression $(x + y)^n$ out in a convenient way: it expands $(x + y)^n$ as

$$\sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Examples:

- $(x + y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2.$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ by applying binomial coefficients.

Binomial coefficients are symmetric because $\binom{n}{k} = \binom{n}{n-k}.$



The Pigeonhole Principle

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The **Pigeonhole Principle** says that if $n + 1$ or more items are placed into n boxes, at least one box contains two or more items.

The principle is often used to prove that something must happen or exist.

Examples:

- If 13 socks are placed into 12 drawers, at least one drawer has at least 2 socks.
- In a group of 367 people, at least two people share the same birthday.

The Pigeonhole Principle is a powerful tool for **existence proofs** (= proofs that show that something exists.)

Many simple but surprising results follow from this basic idea. See some of them in [this article on Medium.com](#).



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Fun Class Activities:

1. Find $\binom{8}{3}$.
2. Expand $(a + b)^3$.
3. Find the coefficient of x^2y^3 in $(x + y)^5$.
4. There are 11 candies in 5 jars. Explain why one jar has at least 3 candies.
5. There are 13 people in a room. Explain why at least two of them share a birth month.
6. There are 5 socks in 2 drawers. Show that one drawer has at least 3 socks.



Any questions?



Counting & Combinatorics: Sources

Aspnes, James. (2022). [Chapter 11](#). *Notes on Discrete Mathematics*. License: [CC BY-SA: Attribution-ShareAlike](#).

“CS202: Discrete Structures | [Unit 2](#).” Saylor Academy. License: [CC BY: Attribution](#).

Davis, Stephen. (2019). [Chapter 8](#). *A Cool Brisk Walk Through Discrete Mathematics*, version 2.2. License: [CC BY-SA: Attribution-ShareAlike](#).

Doerr, Al and Levasseur, Ken. (2024). [Chapter 2](#). *Applied Discrete Structures*, <https://discretemath.org/>. License: [CC BY-NC-SA: Attribution-NonCommercial-ShareAlike](#).

Jamaloodie, Mohamed, et al. [Chapter 9](#). *Discrete Math*. Georgia Gwinnett College. License: [CC BY-NC: Attribution-NonCommercial](#).

Keller, Mitchel T and Trotter, William T. [Chapter 2](#). *Applied Combinatorics*. Open Textbook Library, 2017. License: [CC BY-SA: Attribution-ShareAlike](#).

Levin, Oscar. (2023). [Chapter 1](#). *Discrete Mathematics: An Open Introduction*, <https://discrete.openmathbooks.org/dmoi3/>, 3rd edition. License: [CC BY-SA: Attribution-ShareAlike](#).

Lumen Learning. [College Algebra | Counting Principles](#). License: [CC BY: Attribution](#).

Sassolas, Mathieu. (2021). [Chapter VIII](#). *Introduction to Discrete Mathematics: An OER for MA-471*, https://academicworks.cuny.edu/qb_oers/179/. License: [CC BY-NC: Attribution-NonCommercial](#).



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